MA633L-Numerical Analysis

Lecture 13 : Hermite Interpolation

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Taylor and Lagrange Interpolation

- A Taylor polynomial of degree n interpolates the function. Its first n derivatives at one point.
- A Lagrange polynomial of degree n interpolates the function values at $n + 1$ points.

Can we combine the ideas of Taylor and Lagrange to get an interpolating polynomial that matches both the function values and some number of dertivatives at multiple points?
Polynomials exist! It is called Osculating Polynomials.

Osculating Polynomial Interpolation

Osculating Polynomial Interpolation

Definition 1 (Osculating Polynomial)

- Consider the $n+1$ distinct points $\{x_0, x_1, x_2, \cdots, x_n\} \in [a, b]$, non-negative integers $\{m_0, m_1, m_2, \cdots, m_n\}$ and $m = \max\{m_0, m_1, m_2, \cdots, m_n\}$
- The osculating polynomial interpolation of a function $f \in C^m[a, b]$ at x_i, is the polynomial (of lowest possible order) that agrees with

$$\{f(x_i), f'(x_i), \cdots, f^{(m_i)(x_i)}\} \text{ at } x_i \in [a, b], \forall i$$

- The degree of the osculating polynomial is at most

$$M = n + \sum_{i=0}^{n} m_i$$

Osculating Polynomial

That is, the osculating polynomial that approximats f is the polynomial $P(x)$ of least degree such that

$$f^{(k)}(x_i) = P^{(k)}(x_i), \forall i = 0, 1, 2, \cdots, n, \forall k = 0, 1, 2, \cdots, m_i$$

That is,

$$\begin{aligned}
f(x_0) = P(x_0), f'(x_0) = P'(x_0), f''(x_0) = P''(x_0), \cdots, f^{(m_0)}(x_0) = P^{(m_0)}(x_0), \\
f(x_1) = P(x_1), f'(x_1) = P'(x_1), f''(x_1) = P''(x_1), \cdots, f^{(m_1)}(x_1) = P^{(m_1)}(x_1), \\
\vdots \\
f(x_n) = P(x_n), f'(x_n) = P'(x_n), f''(x_n) = P''(x_n), \cdots, f^{(m_n)}(x_n) = P^{(m_n)}(x_n),
\end{aligned}$$

Hermite Polynomial Interpolation

- If we let $m_i = 1, \forall i$, the polynomial is called a Hermite Polynomial Interpolation.
- If $n = 0$, then it has only single point x_0 and we obtain only Taylor polynomial of m_0^{th} degree.
- What will be the answer when $m_i = 0, \forall i$?
- **Recall:** For given $(x_i, f(x_i)), i = 1, 2, \cdots, n$, we define the Lagrange coefficients as

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)$$

- $\ell_i(x)$ is often denoted as $L_{n,i}(x)$.

Hermite Polynomial Interpolation

Theorem 1 (Theorem on Hermite Polynomial)

If $f \in C^1[a,b]$ and $x_0, x_1, x_2, \cdots, x_n$ are distinct points in $[a,b]$, the unique polynomial of least degree $(\leq 2n+1)$ agreeing with $f(x)$ and $f'(x)$ at $\{x_0, x_1, x_2, \cdots, x_n\}$ is the Hermite polynomial,

$$H_{2n+1}(x) = \sum_{i=0}^{n} f(x_i) H_{n,i}(x) + \sum_{i=0}^{n} f'(x_i) \hat{H}_{n,i}(x)$$

where $H_{n,i}(x) = [1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x)$ and $\hat{H}_{n,i}(x) = (x - x_i)\ell_i^2(x)$.

Moreover, if $f \in C^{2n+2}[a,b]$, then for any $\overline{x} \in [a,b]$, there is a point $\xi(\overline{x}) \in (a,b)$ such that

$$f(\overline{x}) - H_{2n+1}(\overline{x}) = \frac{f^{(2n+2)}(\xi(\overline{x}))}{(2n+2)!} \prod_{i=0}^{n} (\overline{x} - x_i)^2$$

Hermite Polynomial Interpolation

Proof:

- Since

$$\ell_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

  we have

$$H_{n,j}(x_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

  and

$$\hat{H}_{n,j}(x_i) = 0 \;\; \forall i, j$$

Hence,

$$H_{2n+1}(x_i) = f(x_i) \;\; \forall i$$

Hermite Polynomial Interpolation

- Similarly

$$\hat{H}'_{n,j}(x_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

  and

$$H'_{n,j}(x_i) = 0 \ \ \forall i, j$$

Hence,

$$H'_{2n+1}(x_i) = f'(x_i) \ \ \forall i$$

Hermite Polynomial Interpolation

Uniqueness:

Let $K_{2n+1}(x)$ be a polynomial that satisfies the above conditions. That is,

$$K_{2n+1}(x_i) = f(x_i), \forall i$$

and

$$K'_{2n+1}(x_i) = f'(x_i), \forall i$$

Let

$$D(x) = H_{2n+1}(x) - K_{2n+1}(x)$$

Then D is also a polynomial of degree at least $2n+1$

$$D(x_i) = D'(x_i) = 0, \forall i$$

Therefore $D(x)$ has $n+1$ distinct roots of multiplicity 2

Hermite Polynomial Interpolation

Therefore,

$$D(x) = \prod_{i=0}^{n} (x - x_i)^2 Q(x)$$

This is possible only if $Q(x) = 0$. Otherwise, the degree of $D(x)$ is at least $2n + 2$ which is a contradiction. This proves the uniqueness.

Errors in Hermite Polynomial Interpolation

Moreover Part:

If $\overline{x} = x_i$, the proof is obvious, as the equation reduces to zero. Assume that $\overline{x} \neq x_i$. Define a new function $\phi(t)$ in the variable t as follows:

$$\phi(t) = f(t) - H_{2n+1}(t) - \left[\frac{\prod_{i=0}^{n}(t - x_i)^2}{\prod_{i=0}^{n}(\overline{x} - x_i)^2}\right][f(\overline{x}) - H_{2n+1}(\overline{x})]$$

Since x_i's are distinct and $\overline{x} \neq x_i$, the function ϕ is well defined. Also, observe that

$$\phi(x_i) = f(x_i) - H_{2n+1}(x_i) - 0 = 0, \quad 0 \leq i \leq n$$

Errors in Hermite Polynomial Interpolation

Further,

$$\phi(\overline{x}) = f(\overline{x}) - H_{2n+1}(\overline{x}) - \left[\frac{\prod_{i=0}^{n}(t - x_i)^2}{\prod_{i=0}^{n}(\overline{x} - x_i)^2}\right][f(\overline{x}) - H_{2n+1}(\overline{x})] = 0$$

- $\phi = 0$ at the $n+2$ points $x_0, x_1, \cdots, x_n$ and $\overline{x}$.
- Since $f \in C^{n+1}[a, b], P_n \in C^{\infty}[a, b], \phi \in C^{n+1}[a, b]$. Further $\phi(t) = 0$ at the $n+2$ distinct points, therefore as per the Rolle's theorem $\phi'(t)$ has $n+1$ distinct zeros $\xi_0, \xi_1, \cdots, \xi_n$.

Errors in Hermite Polynomial Interpolation

For,

$$\phi'(t) = f'(t) - H'_{2n+1}(t) - \left[\frac{2[f(\overline{x}) - H_{2n+1}(\overline{x})]}{\prod_{i=0}^{n}(\overline{x} - x_i)^2}\right] \sum_{j=0}^{n}(t - x_j) \prod_{\substack{i=0 \\ i \neq j}}^{n}(t - x_i)^2$$

Observe that

$$\phi'(x_i) = f'(x_i) - H'_{2n+1}(x_i) - 0 = 0, \quad 0 \leq i \leq n$$

Errors in Hermite Polynomial Interpolation

- $\phi(t)$ has $2n+2$ zeros in the interval $[a, b]$.
- Since $f \in C^{2n+2}[a, b], H_{2n+1} \in C^{\infty}[a, b], \phi' \in C^{2n+1}[a, b], \phi \in C^{2n+2}[a, b]$.
- Therefore as per the Generalized Rolle's theorem there exists a point $\xi \in (a, b)$ such that $\phi^{(2n+2)}(\xi) = 0$.

$$\phi^{(2n+2)}(\xi) = 0 \implies f^{(2n+2)}(\xi) - H_{2n+1}^{(2n+2)}(\xi) - \frac{d^{2n+2}}{dt^{2n+2}} \left[\frac{\prod_{i=0}^{n} (t - x_i)^2}{\prod_{i=0}^{n} (\overline{x} - x_i)^2} \right]_{t=\xi} [f(\overline{x}) - H_{2n+1}(\overline{x})] = 0$$

Errors in Hermite Polynomial Interpolation

Since H_{2n+1} is a polynomial of degree $\leq 2n+1$, $H_{2n+1}^{(2n+2)}(t) = 0$. Also (Prove!),

$$\frac{d^{2n+2}}{dt^{2n+2}} \left[\frac{\prod_{i=0}^{n} (t - x_i)^2}{\prod_{i=0}^{n} (\overline{x} - x_i)^2} \right] = \frac{(2n+2)!}{\prod_{i=0}^{n} (\overline{x} - x_i)^2}$$

$$\phi^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - \frac{(2n+2)!}{\prod_{i=0}^{n} (\overline{x} - x_i)^2} [f(\overline{x}) - H_{2n+1}(\overline{x})] = 0$$

The proof follows after rearrangements.

Hermite Polynomial Interpolation

Example 2

Suppose that $f(0) = 2, f'(0) = 1, f(1) = 4, f'(1) = 1, f(3) = 5, f'(3) = -2.$ Find the Hermite interpolating polynomial and then find $f(2)$.

$$\ell_0(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}x^2 - \frac{4}{3}x + 1 \qquad \implies \ell_0'(x) = \frac{2}{3}x - \frac{4}{3}$$

$$\ell_1(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x^2 + \frac{3}{2}x \qquad \implies \ell_1'(x) = -x + \frac{3}{2}$$

$$\ell_2(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x^2 - \frac{1}{6}x \qquad \implies \ell_2'(x) = \frac{1}{3}x - \frac{1}{6}$$

Hermite Polynomial Interpolation

$$H_{2,0}(x) = [1 - 2(x-0)\ell_0'(0)]\ell_0^2(x) = \left(1 + \frac{8}{3}x\right)\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2$$

$$H_{2,1}(x) = [1 - 2(x-1)\ell_1'(1)]\ell_1^2(x) = (2 - x)\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2$$

$$H_{2,2}(x) = [1 - 2(x-3)\ell_2'(3)]\ell_2^2(x) = \left(6 - \frac{5}{3}x\right)\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2$$

Hermite Polynomial Interpolation

$$\hat{H}_{2,0}(x) = (x - 0)\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2$$

$$\hat{H}_{2,1}(x) = (x - 1))\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2$$

$$\hat{H}_{2,2}(x) = (x - 3))\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2$$

Hermite Polynomial Interpolation

$$
\begin{aligned}
H_5(x) &= f(0)H_{2,0}(x) + f(1)H_{2,1}(x) + f(3)H_{2,2}(x) \\
&\quad + f'(0)\hat{H}_{2,0}(x) + f'(1)\hat{H}_{2,1}(x) + f'(3)\hat{H}_{2,2}(x) \\
&= \frac{1}{12}(24 + 12x + 57x^2 - 70x^3 + 29x^4 - 4x^5) \\
f(2) \approx H_5(2) &= \frac{13}{3}
\end{aligned}
$$

Challenging Problem: NIRF Data

2022	IITM	IITD	IITB	IITK	IITKGP	IITR	IITG	NITT	IITH	NITK	VIT
NT	5699	6863	5749	5391	10064	5938	4913	4734	1932	4118	36432
NE	6272	7024	6617	5498	9604	5820	4687	5014	1876	4944	33032
NP	2204	3332	2411	1965	2408	2344	1802	967	682	934	2336
SS	18.5	18.5	18.5	18.5	19.31	18.23	16.45	17.5	11.28	16	18.6
Pred SS	18.5	18.5	18.5	18.5	19.3144	18.2317	16.4480	17.5	11.2826	16	18.6001
2023											
NT	6001	7639	7403	5866	11598	6469	5176	5036	2291	4118	37232
NE	6643	7597	8444	5871	10403	6043	5567	5336	2203	5245	40398
NP	2435	3744	4048	2170	3534	2667	2621	903	728	928	2152
SS	18.5	18.43	18.5	18.5	18.45	17.61	18.5	17.5	12.65	17.5	20
Pred SS	18.5	18.4258	18.5	18.5	18.4545	17.6110	18.5	17.5	12.6543	17.5	20
2024											
NT	6355	8026	7705	6201	11230	6896	5605	5202	2615	4118	38214
NE	6612	8296	8531	6123	11051	6465	5530	5411	2444	5553	43299
NP	2574	3807	3987	2057	3676	2764	1994	956	800	1120	2978
SS	18.5	18.5	18.5	18.33	19.76	17.66	18.32	17.5	12.41	17.5	20
Pred SS	18.5	18.5	18.5	18.3302	19.7609	17.6563	18.3194	18.5	12.4115	17.5	20

Here SS is given by NIRF and Pred SS is calculated by me.

Challenging Problem

NIRF Formula for SS

$$SS = 15 f_1(NT, NE) + 5 f_2(NP)$$

where

- NT: Total number of Sanctioned Approved Intake in UG and PG
- NE: Total number of enrolled students in UG and PG
- NP: Total number of students enrolled for the doctoral program till previous academic year.

The function f_1 and f_2 are unknown. How do you find the the SS value for IITT if you know NT,NE and NP for 2025. Can we use ML? Can we use interpolation?

Thanks

Doubts and Suggestions

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