MA633L-Numerical Analysis

Lecture 15 : Solution of Nonlinear Equations: Closed Methods

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February 17, 2025

Nonlinear Equations: Examples

Nonlinear Equations

School Mathematics:

$$ax^2 + bx + c = 0$$

School Physics:

$$s = ut + \frac{1}{2}at^2$$

Nonlinear Equations

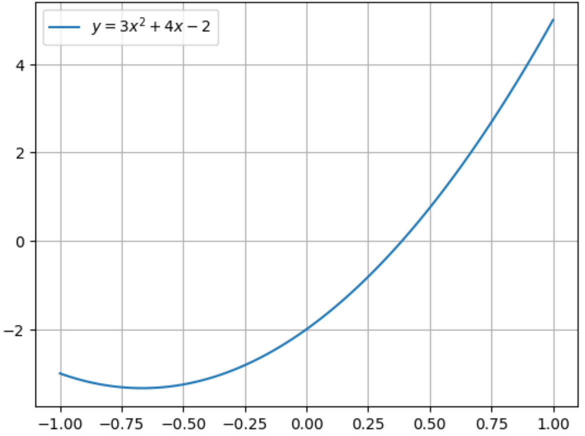


Figure 1: Nonlinear Equations

Nonlinear Equations

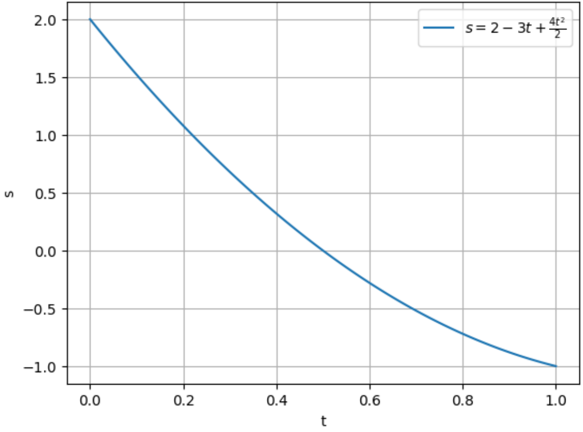


Figure 2: Nonlinear Equations

Nonlinear Equations

Medical studies have established that a bungee jumper's chances of sustaining a significant vertebrae injury increase significantly if the free-fall velocity exceeds 36 m/s after 4 s of free fall.
From a bungee-jumping study, it was identified that the fall velocity of a bungee-jumper is given by

$$v(t) = \sqrt{\frac{gm}{c}} \tanh\left(\sqrt{\frac{gc}{m}} t\right)$$

where m is the mass and c is the drag coefficient.
A bungee-jumping company requested you to to determine the mass at which this criterion is exceeded given a drag coefficient of $0.25 kg/m$

Nonlinear Equations

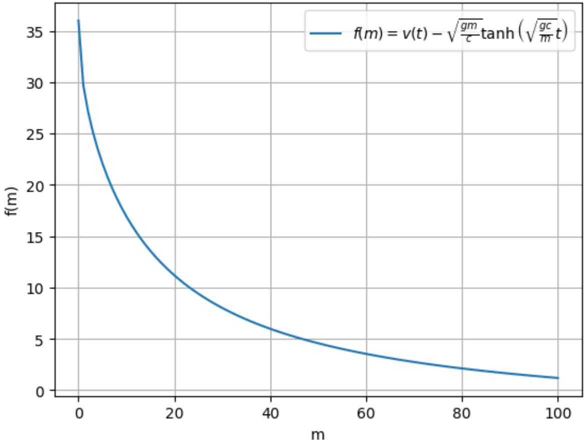


Figure 3: Nonlinear Equations

Nonlinear Equations

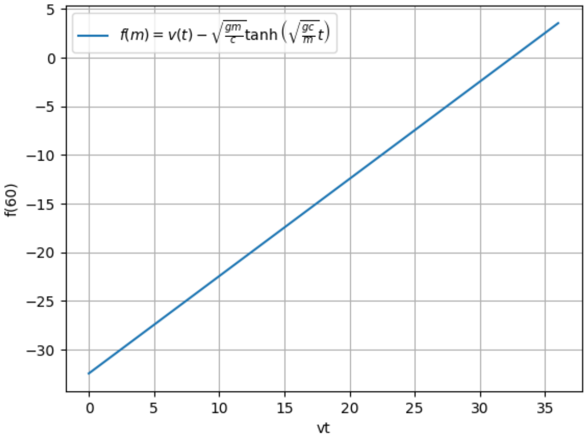


Figure 4: Nonlinear Equations

Nonlinear Equations

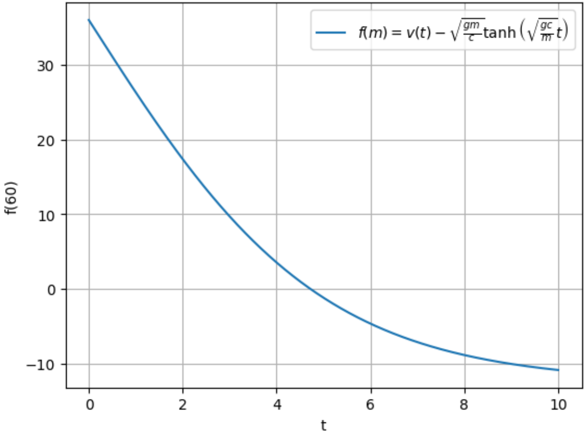


Figure 5: Nonlinear Equations

Nonlinear Equations

The saturation concentration of dissolved oxygen in freshwater can be calculated with the equation

$$\ln o_{sf} = -139.34411 + \frac{1.575701 \times 10^5}{T_a} - \frac{6.642308 \times 10^7}{T_a^2} + \frac{1.243800 \times 10^{10}}{T_a^3} - \frac{8.621946 \times 10^{11}}{T_a^4}$$

where o_{sf} is the saturation concentration of dissolved oxygen freshwater at 1 atm and T_a is the absolute temperature (K). Given a value of oxygen concentration, this formula can be used to solve for temperature in $^\circ C$

Nonlinear Equations

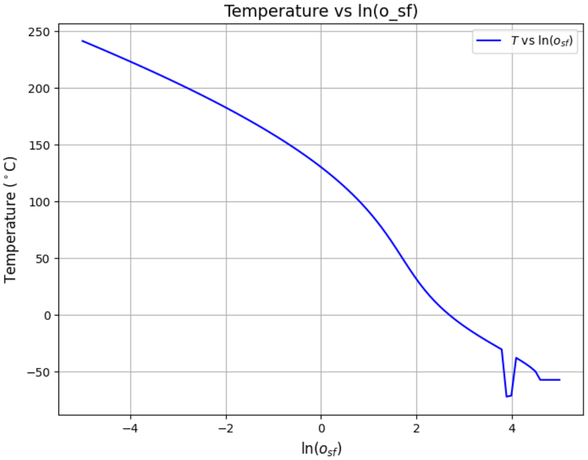


Figure 6: Nonlinear Equations

Nonlinear Equations

A civil engineer as a city planning commission member along with transportation engineers would like to find the population growth trends of a city and adjacent suburb to design the road construction and expanding the city limits. The population of the suburb area is declining with time according to

$$P_s(t) = P_{s,max}e^{-k_s t} + P_{s,min}$$

whereas the city population is growing

$$P_c(t) = \frac{P_{c,max}}{1 + [P_{c,max}/P_0 - 1]e^{-k_c t}}$$

where $P_{s,max}$, $P_{c,max}$, $P_{s,min}$, P_0, k_c, k_u are parameters estimated from the previous population data. If $P_{s,max} = 8,00,000, k_s = 0.05/yr, k_c = 0.09/yr, P_{s,min} = 11,00,000, P_{c,max} = 32,00,000, P_0 = 1,00,000$ find the time when the suburb area population is 20% larger than the city population.

Nonlinear Equations

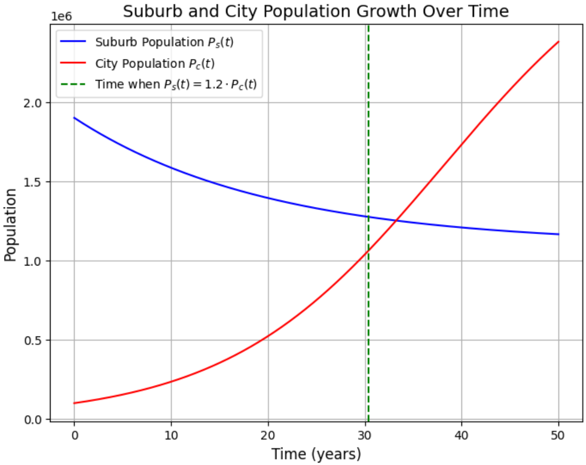


Figure 7: Nonlinear Equations

Nonlinear Equations

A total charge Q is uniformly distributed around a ring-shaped conductor with radius a. A charge q is located at a distance x from the center of the ring. The force exerted on the charge by the ring is given by

$$F = \frac{1}{4\pi e_0} \frac{qQx}{(x^2 + a^2)^{3/2}}$$

where $e_0 = 8.9 \times 10^{-12} C^2/(Nm^2)$. Find the distance x when the force is $1.25N$ if $q = Q = 2 \times 10^{-5} C$ for a ring with a radius $a = 0.85m$.

Nonlinear Equations

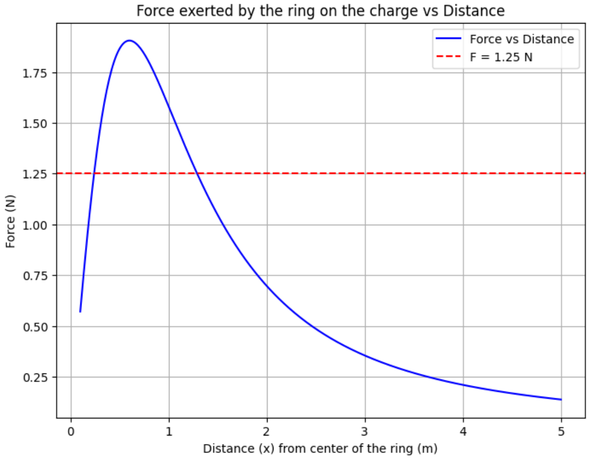


Figure 8: Nonlinear Equations

Nonlinear Equations

The activation function of a node in an artificial neural network is a function that calculates the output of the node based on its individual inputs and their weights. One of the historically important development of neural networks is the sigmoid function. It is one of the popular activation function in early neural networks because the gradient is strongest when the unit's output is near 0.5 and that allows efficient backpropagation training. The sigmoid function is given by

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

What should be the input signal for the sigmoid function so that $\sigma(z) = 0.5$.

Nonlinear Equations

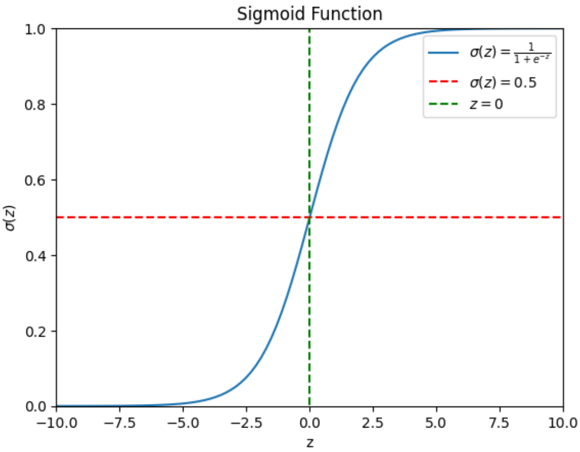


Figure 9: Nonlinear Equations

Nonlinear Equations

For finding equilibrium conditions, or finding eigen values of a system such as characteristic equations

$$x^3 + x = 1.$$

Zeros of Bessel functions:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1)!} \left(\frac{z}{2}\right)^{2k}$$

Nonlinear Equations

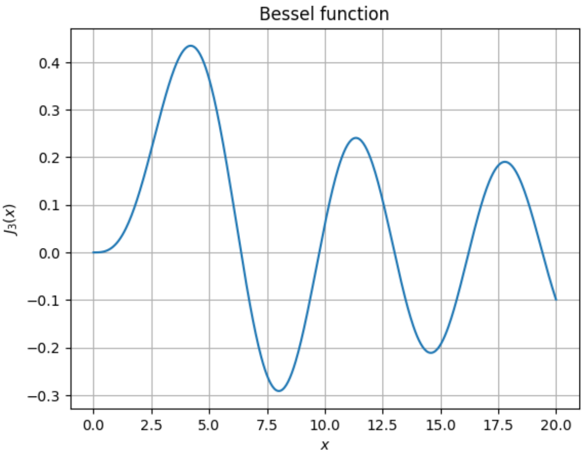


Figure 10: Nonlinear Equations

Nonlinear Equations: Classification

Nonlinear Equations

A function $y = f(x)$ is said to be **linear** if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. A function is said to be **nonlinear** if it is not linear.
A function $y = f(x)$ is **Algebraic** if it can be expressed in the form

$$P_n y^n + P_{n-1} y^{n-1} + \cdots + P_1 y + P_0 = 0$$

where P_k's are polynomial in x. Obviously all polynomials are algebraic functions.
A **Transcendental function** is the one that is not algebraic.

Nonlinear Equations

Examples for Algebraic Functions:

$$\sqrt{x}, x^3 + 1, \frac{\sqrt{1+x^3}}{x^{3/7} - \sqrt{7}x^{1/3}}, \sqrt{\frac{1-x^2}{1+x^2}}, \frac{x^{1/2} - \frac{1}{3}a}{x^{1/3} - x^{1/2}}$$

Examples for Transcendental Functions:

$$\sin x - 0.5x, \tan x - x, \cosh x - \sec x, \cosh x \cos x - -1, \ln x^2 - 2$$

Condition Number of a Function

Ill and Well-Conditioned Problem

Definition 1 (Well-Conditioned Problem)
A problem is well conditioned if a small change in the input (the data) always creates a small change in the output (the solution).

Definition 2 (Ill-Conditioned Problem)
A problem is ill-conditioned if a small change in the input (data) can create a large change in the output (solution).

Condition Number

Definition 3 (Condition Number)
Size of Change in Solution $=$ condition number $\times$ size of change in input

If we have measure to measure the size, say $||.||$

Definition 4 (Absolute Condition Number (ν))
$||$ Change in Solution $|| = \nu \times ||$ Change in Input $||$

Definition 5 (Relative Condition Number (κ))

$$\frac{||\text{Change in Solution}||}{||\text{Solution}||} = \kappa \times \frac{||\text{Change in Input}||}{||\text{Input}||}$$

Condition Number of a function

Assume that $f \in C^1[a,b]$. Let x be the input and δ be the change in input. Then the output is $f(x)$ and the change in output is $f(x+\delta)$. By Taylor's theorem

$$f(x+\delta) = f(x) + \delta f'(x) + O(\delta^2)$$

$$\implies |f(x+\delta) - f(x)| \approx |f'(x)||\delta|$$

Therefore,

Definition 6 (Absolute Condition Number (ν))

$$\nu = |f'(x)|$$

Condition Number of a function

Further,

$$\frac{|f(x+\delta) - f(x)|}{|f(x)|} \approx \left|\frac{xf'(x)}{f(x)}\right| \left|\frac{\delta}{x}\right|$$

Definition 7 (Relative Condition Number (κ))

$$\kappa = \left|\frac{xf'(x)}{f(x)}\right|$$

If $f \in C^1(\mathbb{R}^n)$, it is defined as

$$\kappa = \frac{\|x\| \|J(x)\|}{\|f(x)\|}$$

where $J(x)$ denotes the Jacobian matrix, $x \in \mathbb{R}^n$.

Nonlinear Equations Solvers: Classifications

Nonlinear Equations

In general, finding the equilibrium of a function or root of an equation, can be rewritten as finding x such that

$$f(x) = 0.$$

The function or equation may be linear or nonlinear, however, there is no general formula to find the exact solution of nonlinear equations.
For example, $f(x) = e^{-x} - x$ cannot be solved analytically.
In such scenarios, the only alternative is to find an approximate solution.

Nonlinear Equations

The easiest way to obtain an approximate solution is to plot the function and find where it crosses the x-axis.
$f(x) = 0$ for some value of x, then x is called as a **root**.
Other than this, we can use trial and error method, where we should guess values until we obtain $f(x) = 0$.

Condition Number of Root finding

Definition 8 (Absolute Condition Number of root finding (ν_r))

$$\nu_r = \left|\frac{1}{f'(x)}\right|$$

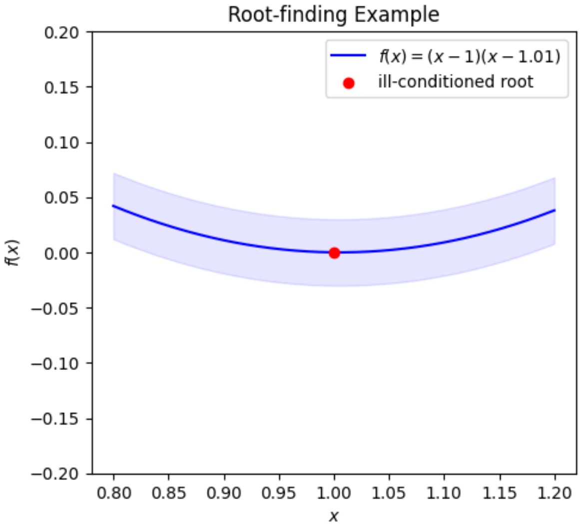
Definition 9 (Relative Condition Number (κ_r))

$$\kappa = \left|\frac{f(x)}{x f'(x)}\right|$$

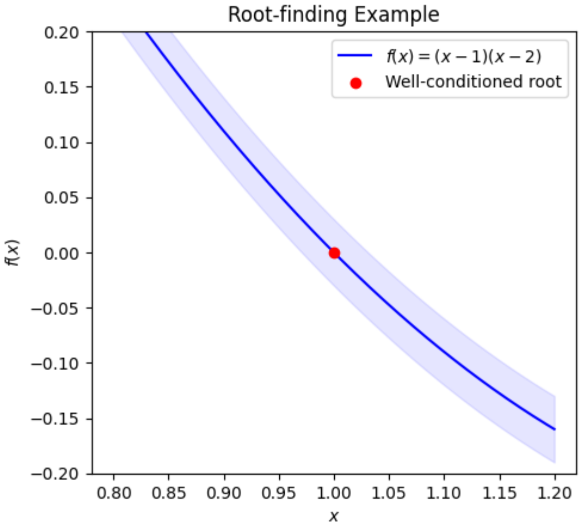
Condition Number of Root finding

- If $f'(r) = 0$, then $\nu_r = \infty, \kappa_r = \infty$
- Alternatively, finding ν_r and κ_r is same as finding ν and κ for f^{-1} at 0.
- When $|f'|$ is so small near the root, you may get a large error in the root estimate for the inverse problem and it is ill-conditioned inverse problem.
- On the other hand, if $|f'|$ is too large x, it is ill-conditioned forward problem.

Nonlinear Equations



Nonlinear Equations



Nonlinear Equations

The roots of equations may be either real or complex. Finding the roots or locating the roots are primarily divided into two areas:

1. Locating a single real root of algebraic and transcendental equations from the knowledge of its approximate location
2. Locating all real and complex roots of polynomials.

Nonlinear Equations

Further, we subdivide finding the roots of a nonlinear equation into

1. **bracketing methods** for which we start with guesses that bracket or contain the root and then systematically reduce the width of the bracket
2. **open method** for which trial-and-error iterations are required without initial guesses to bracket the root.

Open methods are usually more efficient (computationally) than bracketing method, however, it is not guaranteed to bracket the root.

Nonlinear Equations

The basic idea of bracketing method is applying the intermediate value theorem.

Theorem 1

Intermediate Value Theorem Suppose $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then a number $c \in (a, b)$ exists with $f(c) = K$.

In particular, if $f(a) > 0$ and $f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$. As the name implies, the guesses must bracket the root.

Bisection Method

Bisection Method

- If $f \in C[a_0, b_0]$ with $f(a_0)$ and $f(b_0)$ of opposite sign, that is,

$$f(a_0)f(b_0) < 0,$$

  then there is at least one real root x_0 lies between a_0 and b_0, where a_0 and b_0 are guesses made to bracket the root.
- Our goal is to find x_r. As an incremental search, divide the interval into half, that is, find whether

$$f(a_0)f\left(\frac{a_0 + b_0}{2}\right) < 0.$$

- If so, then the root is in the left half of the interval, otherwise the root is in the right half of the interval.
- The Algorithm 1 depicts the bisection method.

Bisection Method

- We are bisecting the interval at every iteration, therefore, it is named as bisection method.
- Since we are dividing the interval by half every time, this method is also called interval **halving method.**
- As we are chopping one part of the interval, it is also called **chopping method.**
- Since, the root lies on either side of half of the interval, it is also called as **binary method.**
- Also, it is named after Bolzano and hence **Bolzano's method.**

Bisection Method

The present approximation relative error (ϵ_a) and the true relative error (ϵ_t) is given by:

$$\epsilon_a = \left| \frac{x_r^{n-1} - x_r^n}{x_r^n} \right| \quad \text{and} \quad \epsilon_t = \left| \frac{TV - x_r^n}{TV} \right|$$

Here, TV denotes the True Value.

Bisection Method

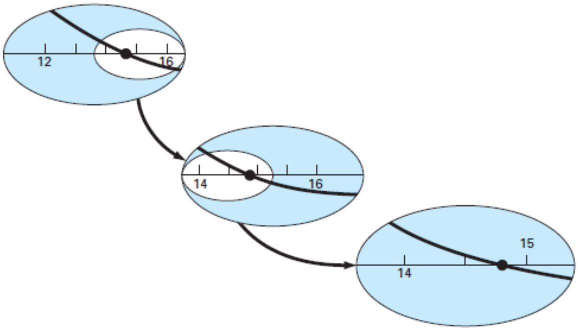


Figure 13: Nonlinear Equations

Figure 14: Bisection

Bisection Animation

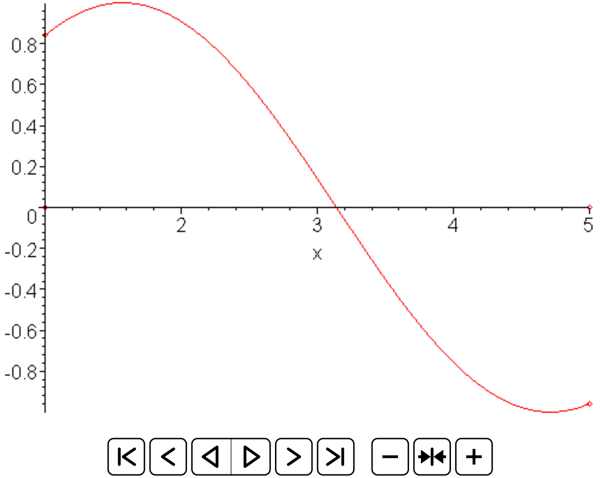


Figure 15: Nonlinear Equations

Bisection Method

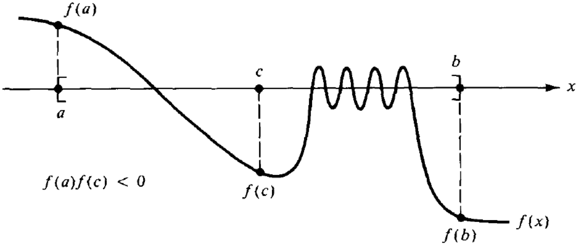


Figure 16: Bisection, Source: Kincaid Book

Bisection Method

Example 2

Consider the function $f(x) = 6x^2 - 13x + 6$ and its factorization is $(2x-3)(3x-2)$. Therefore, obviously the roots are 1.5 and 2/3. If $a_0 = 1, b_0 = 3$, then for the same equation, we obtain the following table.

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	1.000	3.000	2.000		0.333	4.000
1	1.000	2.000	1.500	0.333	0.000	0.000

The root is 1.500000.

Bisection Method

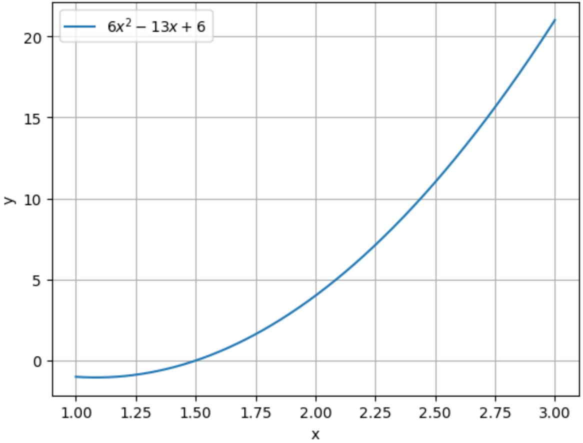


Figure 17: Bisection

Bisection Method

Example 3

If we consider the $a_0 = 0, b_0 = 1$ in the bisection method, then it should print the following table.

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	0.000000	1.000000	0.500000		0.250000	1.000000
1	0.500000	1.000000	0.750000	0.333333	0.125000	0.375000
2	0.500000	0.750000	0.625000	0.200000	0.062500	0.218750
3	0.625000	0.750000	0.687500	0.090909	0.031250	0.101562
4	0.625000	0.687500	0.656250	0.047619	0.015625	0.052734
5	0.656250	0.687500	0.671875	0.023256	0.007813	0.025879
6	0.656250	0.671875	0.664062	0.011765	0.003906	0.013062
7	0.664062	0.671875	0.667969	0.005848	0.001953	0.006500
8	0.664062	0.667969	0.666016	0.002933	0.000977	0.003258
9	0.666016	0.667969	0.666992	0.001464	0.000488	0.001627
10	0.666016	0.666992	0.666504	0.000733	0.000244	0.000814
11	0.666504	0.666992	0.666748	0.000366	0.000122	0.000407
12	0.666504	0.666748	0.666626	0.000183	0.000061	0.000203
13	0.666626	0.666748	0.666687	0.000092	0.000031	0.000102
14	0.666626	0.666687	0.666656	0.000046	0.000015	0.000051
15	0.666656	0.666687	0.666672	0.000023	0.000008	0.000025
16	0.666656	0.666672	0.666664	0.000011	0.000004	0.000013
17	0.666664	0.666672	0.666668	0.000006	0.000002	0.000006

The root is 0.666656.

Bisection Method

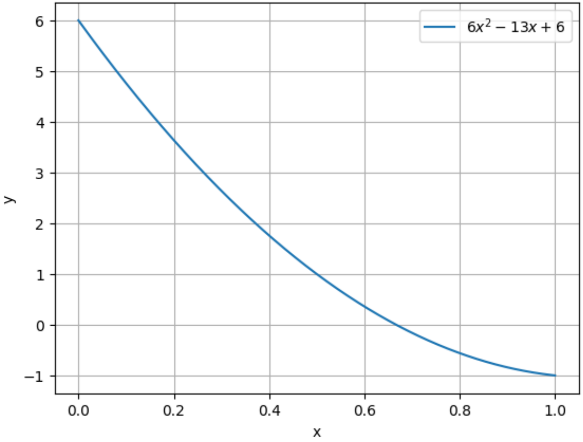


Figure 18: Bisection

Bisection Animation

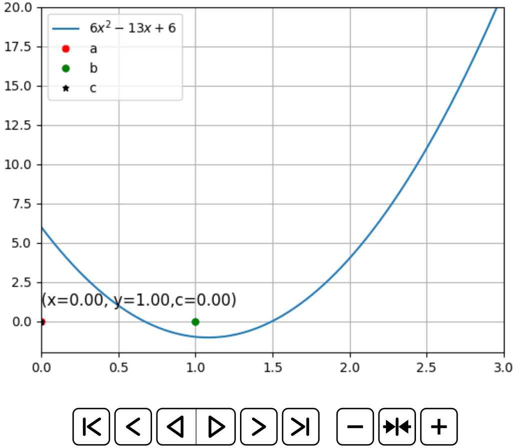


Figure 19: Nonlinear Equations

Bisection Method

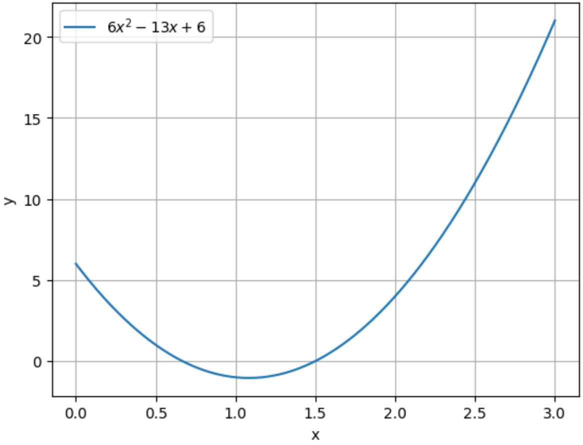


Figure 20: Bisection

Bisection Animation

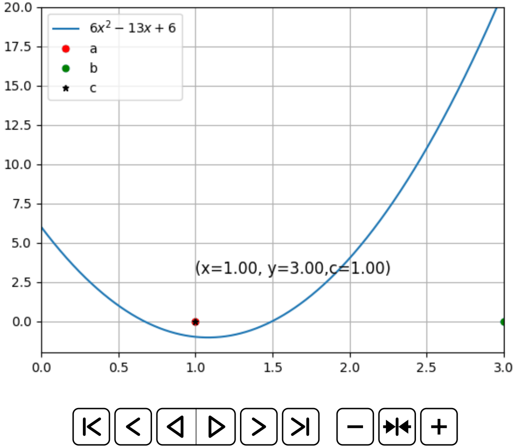


Figure 21: Nonlinear Equations

Bisection Method

Example 4

Use Bisection method to find the solution of the following equation

$$\frac{668.06(1-e^{-0.146843x})}{x} = 40$$

with $a_0 = 12, b_0 = 16, TV = 14.8011$

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	12.000000	16.000000	14.000000		0.054124	1.611347
1	14.000000	16.000000	15.000000	0.066667	0.013438	0.384265
2	14.000000	15.000000	14.500000	0.034483	0.020343	0.593910
3	14.500000	15.000000	14.750000	0.016949	0.003452	0.100032
4	14.750000	15.000000	14.875000	0.008403	0.004993	0.143300
5	14.750000	14.875000	14.812500	0.004219	0.000770	0.021931
6	14.750000	14.812500	14.781250	0.002114	0.001341	0.038976
7	14.781250	14.812500	14.796875	0.001056	0.000285	0.008504
8	14.796875	14.812500	14.804688	0.000528	0.000242	0.006719
9	14.796875	14.804688	14.800781	0.000264	0.000022	0.000891
10	14.800781	14.804688	14.802734	0.000132	0.000110	0.002914
11	14.800781	14.802734	14.801758	0.000066	0.000044	0.001011
12	14.800781	14.801758	14.801270	0.000033	0.000011	0.000060
13	14.800781	14.801270	14.801025	0.000016	0.000005	0.000416

The root is 14.8011.

Bisection Method

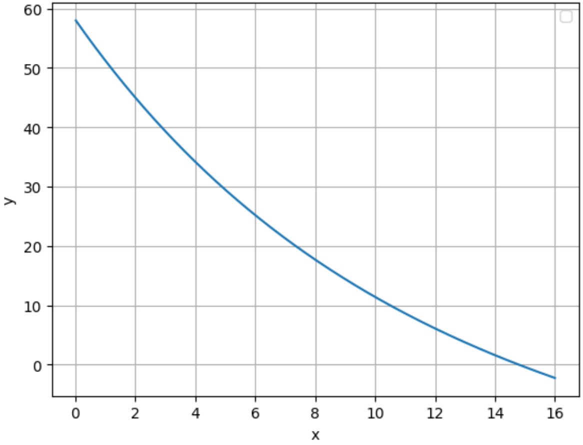


Figure 22: Bisection

Bisection Animation

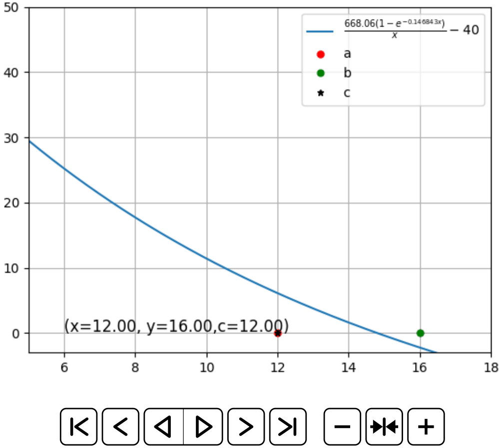


Figure 23: Nonlinear Equations

Bisection Method

- In practice, when we apply bisection algorithm, the computation may not terminate as finding an x such that $f(x) = 0$ may not be possible, however, it is possible to find x such that $|f(x)| < \epsilon$.
- Hence, we terminate the computation when the value of $|f(x)|$ is as small as required.
- Also, we have used the present relative error ϵ_a also as other stopping criteria.
- Although, the machine can represent a double as 2.2250×10^{-308}, which is 2^{-1022}, we could consider maximum number of iterations as 1000 so that the interval can be halved up to 10^{-302}, which is evident from the next theorem.
- For all example problems discussed in this chapter and next chapter, we have used $M = 1000$ or $\epsilon_s = 10^{-5}$ or $\epsilon = 10^{-6}$ as stopping criteria.

6

Thanks

Doubts and Suggestions

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February 17, 2025

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