MA633L-Numerical Analysis

Lecture 16 : Solution of Nonlinear Equations: Bracketing Methods

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February 13, 2025

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Bisection Method

Bisection Method

- If $f \in C[a_0, b_0]$ with $f(a_0)$ and $f(b_0)$ of opposite sign, that is,

$$f(a_0)f(b_0) < 0,$$

  then there is at least one real root x_0 lies between a_0 and b_0, where a_0 and b_0 are guesses made to bracket the root.
- Our goal is to find x_r. As an incremental search, divide the interval into half, that is, find whether

$$f(a_0)f\left(\frac{a_0 + b_0}{2}\right) < 0.$$

- If so, then the root is in the left half of the interval, otherwise the root is in the right half of the interval.
- The Algorithm 1 depicts the bisection method.

Bisection Method

- We are bisecting the interval at every iteration, therefore, it is named as bisection method.
- Since we are dividing the interval by half every time, this method is also called interval **halving method.**
- As we are chopping one part of the interval, it is also called **chopping method.**
- Since, the root lies on either side of half of the interval, it is also called as **binary method.**
- Also, it is named after Bolzano and hence **Bolzano's method.**

Bisection Method

The present approximation relative error (ϵ_a) and the true relative error (ϵ_t) is given by:

$$\epsilon_a = \left|\frac{x_r^{n-1} - x_r^n}{x_r^n}\right| \quad \text{and} \quad \epsilon_t = \left|\frac{TV - x_r^n}{TV}\right|$$

Here, TV denotes the True Value.

Bisection Method

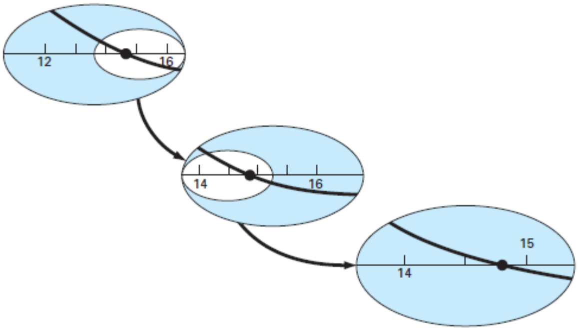

Figure 1: Bisection

Bisection Animation

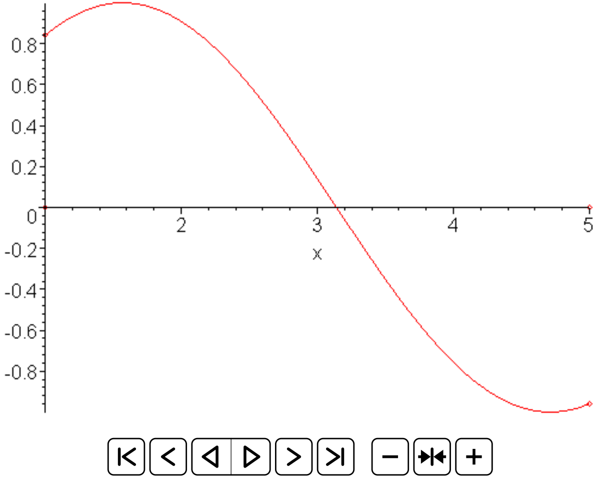

Figure 2: Nonlinear Equations

Bisection Method

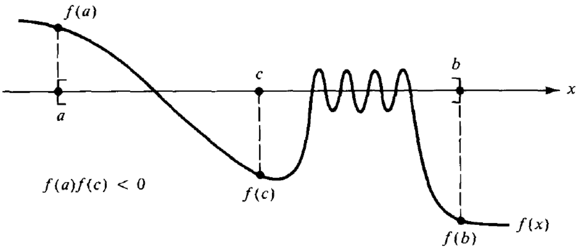

Figure 3: Bisection, Source: Kincaid Book

Bisection Method

Example 1

Consider the function $f(x) = 6x^2 - 13x + 6$ and its factorization is $(2x-3)(3x-2)$. Therefore, obviously the roots are 1.5 and 2/3. If $a_0 = 1, b_0 = 3$, then for the same equation, we obtain the following table.

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	1.000	3.000	2.000		0.333	4.000
1	1.000	2.000	1.500	0.333	0.000	0.000

The root is 1.500000.

Bisection Method

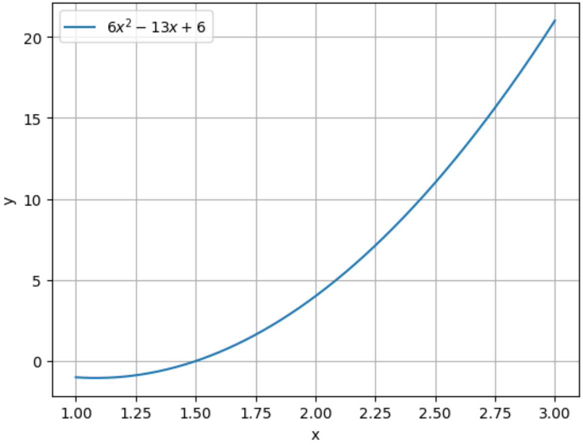

Figure 4: Bisection

Bisection Method

Example 2

If we consider the $a_0 = 0, b_0 = 1$ in the bisection method, then it should print the following table.

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	0.000000	1.000000	0.500000		0.250000	1.000000
1	0.500000	1.000000	0.750000	0.333333	0.125000	0.375000
2	0.500000	0.750000	0.625000	0.200000	0.062500	0.218750
3	0.625000	0.750000	0.687500	0.090909	0.031250	0.101562
4	0.625000	0.687500	0.656250	0.047619	0.015625	0.052734
5	0.656250	0.687500	0.671875	0.023256	0.007813	0.025879
6	0.656250	0.671875	0.664062	0.011765	0.003906	0.013062
7	0.664062	0.671875	0.667969	0.005848	0.001953	0.006500
8	0.664062	0.667969	0.666016	0.002933	0.000977	0.003258
9	0.666016	0.667969	0.666992	0.001464	0.000488	0.001627
10	0.666016	0.666992	0.666504	0.000733	0.000244	0.000814
11	0.666504	0.666992	0.666748	0.000366	0.000122	0.000407
12	0.666504	0.666748	0.666626	0.000183	0.000061	0.000203
13	0.666626	0.666748	0.666687	0.000092	0.000031	0.000102
14	0.666626	0.666687	0.666656	0.000046	0.000015	0.000051
15	0.666656	0.666687	0.666672	0.000023	0.000008	0.000025
16	0.666656	0.666672	0.666664	0.000011	0.000004	0.000013
17	0.666664	0.666672	0.666668	0.000006	0.000002	0.000006

The root is 0.666656.

Bisection Animation

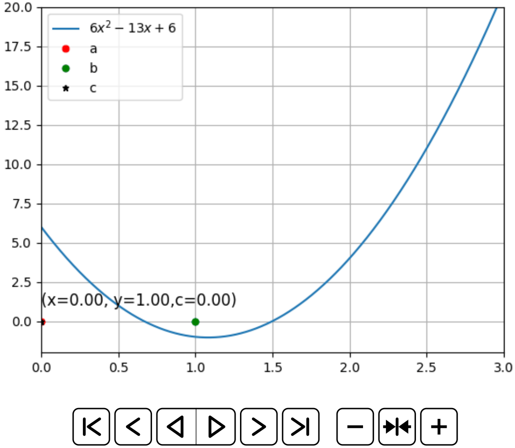

Figure 5: Nonlinear Equations

Bisection Method

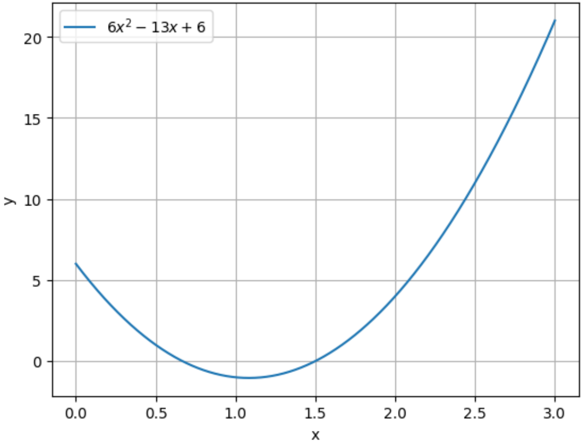

Figure 6: Bisection

Bisection Animation

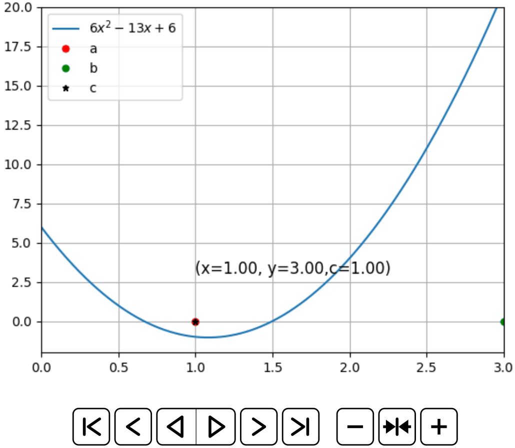

Figure 7: Nonlinear Equations

Bisection Method

Example 3

Use Bisection method to find the solution of the following equation

$$\frac{668.06(1-e^{-0.146843x})}{x} = 40$$

with $a_0 = 12, b_0 = 16, TV = 14.8011$

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	12.000000	16.000000	14.000000		0.054124	1.611347
1	14.000000	16.000000	15.000000	0.066667	0.013438	0.384265
2	14.000000	15.000000	14.500000	0.034483	0.020343	0.593910
3	14.500000	15.000000	14.750000	0.016949	0.003452	0.100032
4	14.750000	15.000000	14.875000	0.008403	0.004993	0.143300
5	14.750000	14.875000	14.812500	0.004219	0.000770	0.021931
6	14.750000	14.812500	14.781250	0.002114	0.001341	0.038976
7	14.781250	14.812500	14.796875	0.001056	0.000285	0.008504
8	14.796875	14.812500	14.804688	0.000528	0.000242	0.006719
9	14.796875	14.804688	14.800781	0.000264	0.000022	0.000891
10	14.800781	14.804688	14.802734	0.000132	0.000110	0.002914
11	14.800781	14.802734	14.801758	0.000066	0.000044	0.001011
12	14.800781	14.801758	14.801270	0.000033	0.000011	0.000060
13	14.800781	14.801270	14.801025	0.000016	0.000005	0.000416

The root is 14.8011.

Bisection Method

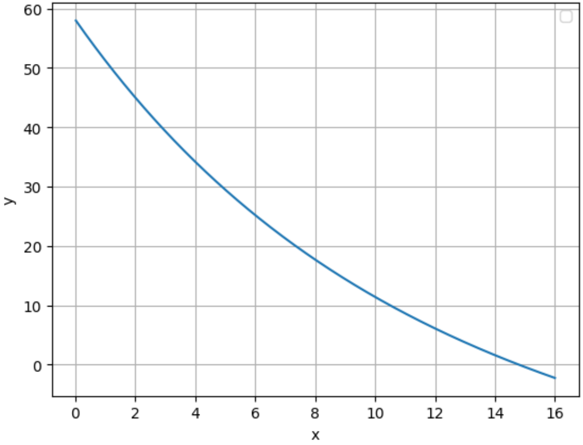

Figure 8: Bisection

Bisection Animation

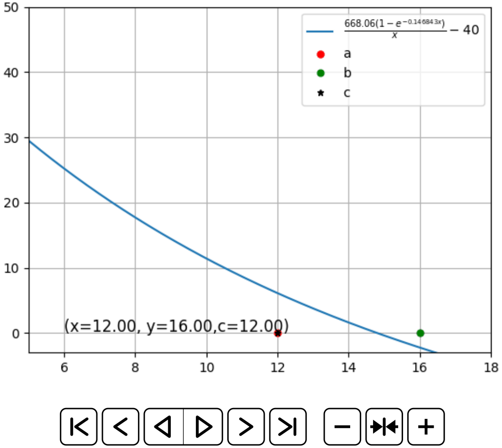

Figure 9: Nonlinear Equations

Bisection Method

- In practice, when we apply bisection algorithm, the computation may not terminate as finding an x such that $f(x) = 0$ may not be possible, however, it is possible to find x such that $|f(x)| < \epsilon$.
- Hence, we terminate the computation when the value of $|f(x)|$ is as small as required.
- We can also use the present relative error ϵ_a also as other stopping criteria.
- Although, the machine can represent a smallest number in double as 2.2250×10^{-308}, which is 2^{-1022}, we could consider maximum number of iterations as 1000 so that the interval can be halved up to 10^{-302}, which is evident from the next theorem.
- For all example problems discussed in this part, we have used $M = 1000$ or $\epsilon_s = 10^{-5}$ or $\epsilon = 10^{-6}$ as stopping criteria.

Bisection Method: Convergence

Bisection Method

Theorem 4

Let $f \in C[a,b]$ be such that $f(a)f(b) < 0$. If the bisection algorithm is applied on f, then after n steps, an approximate root will have been computed with error at most

$$\frac{b-a}{2^{n+1}}$$

Proof:

Let $a_0 = a, b_0 = b, I_0 = [a_0, b_0]$. Suppose f is continuous and $f(a_0)f(b_0) < 0$, then by intermediate value theorem there is a root $r \in [a_0, b_0]$. Let x_r be a root of f and

$$x_0^r = \frac{a_0 + b_0}{2}$$

Then

$$|x_0^r - x_r| \leq \frac{b_0 - a_0}{2}$$

Bisection Method

Now, if the bisection algorithm is applied to compute $a_1, b_1, x_1^r, a_2, b_2, x_2^r$ and so on, then we obtain that

$$|x_n^r - x_r| \leq \frac{b_n - a_n}{2}$$

From the iterative process, it is immediately follows that,

$$\frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}} \implies |x_n^r - x_r| \leq \frac{b_0 - a_0}{2^{n+1}}$$

Hence the proof.

Bisection Method

Note that,

$$\frac{|x_{n+1}^r - x_r|}{|x_n^r - x_r|^\alpha} = \frac{b_0 - a_0}{2^{n+2}} \times \left(\frac{2^{n+1}}{b_0 - a_0}\right)^\alpha = \frac{1}{2}\left(\frac{2^{n+1}}{b_0 - a_0}\right)^{\alpha - 1}$$

The above sequence converges if $\alpha = 1$. Also $C = \frac{1}{2} \implies$ Bisection method has at least linear convergence with rate of convergence $\frac{1}{2}$.

Regula-Falsi Method

Regula-Falsi Method

- The False-Position or Regula-Falsi method has the same features as the bisection method.
- That is, it brackets the roots in a sequence of intervals of decreasing size.
- However, instead of selecting the the midpoint of each interval, this method uses the point where the secant lines intersect the $x-$ axis.

Regula-Falsi Method

The secant line over the interval $[a, b]$ is the chord between $(a, f(a))$ and $(b, f(b))$.
The line joining of these two points and its intersection with $x-$ axis is given by

$$\frac{y - f(b)}{f(a) - f(b)} = \frac{x - b}{a - b} \quad \text{and} \quad y = 0 \implies x = b - f(b)\frac{a - b}{f(a) - f(b)}$$

- Guessing the bracketing values a_0, b_0 and using the above equation brackets the root by the below algorithm.
- The major difference between the false-position method and the bisection method is only the choice of subdivision of intervals.

Regula-Falsi Method

1. Check the condition $f(a_0)f(b_0) < 0$
2. Estimate the root $x_0^r = b_0 - f(b_0)\frac{(a_0 - b_0)}{f(a_0) - f(b_0)}$
3. Make the following evaluations to determine in which subinterval the root lies:
4. If $f(a_0)f(x_0^r) < 0$, the root lies in the lower subinterval, set $b_0 = x_0^r$, go to step 2
5. If $f(a_0)f(x_0^r) > 0$, the root lies in the upper subinterval, set $a_0 = x_0^r$, go to step 2
6. If $f(a_0)f(x_0^r) = 0$, the root is x_0^r, terminate the computation

Regula-Falsi Animation

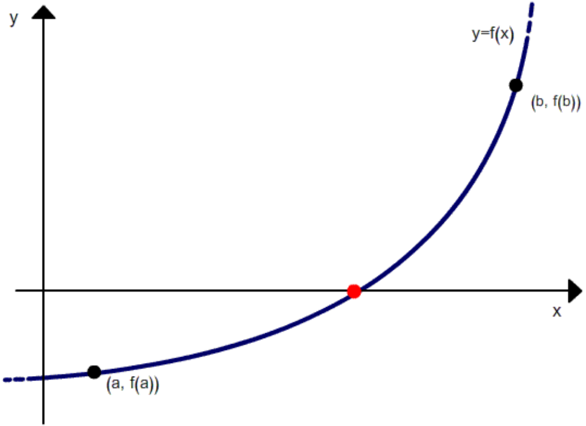

Figure 10: Regula-Falsi

Regula-Falsi Method

Example 5

The velocity v of a falling parachutist is given by

$$v = \frac{gm}{c}(1 - e^{-(c/m)t})$$

where $g = 9.81m/s^2$, c drag coefficient and m mass of the parachutist. Use False-position method to find the drag coefficient c needed for a parachutist of mass $m = 68.1kg$ to have a velocity of $40m/s$ after free falling for time $t = 10s$. This problem can be solved by finding the root of the following equation.

$$\frac{668.06(1 - e^{-0.146843x})}{x} = 40$$

with $a_0 = 12, b_0 = 16, TV = 14.8011$

Regula-Falsi Method

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	12.00	16.000000	14.930968		0.008774	0.251481
1	12.00	14.930968	14.815179	0.007816	0.000951	0.027146
2	12.00	14.815179	14.802736	0.000841	0.000111	0.002916
3	12.00	14.802736	14.801399	0.000090	0.000020	0.000313
4	12.00	14.801399	14.801256	0.000010	0.000011	0.000034

The root is 14.801127. But, as per the Bisection algorithm, it takes more number of steps.

Regula-Falsi Animation

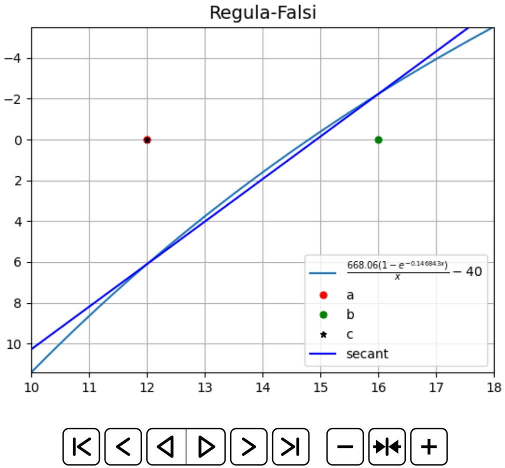

Figure 11: Regula-Falsi

Regula-Falsi Method

Example 6

Consider the function $f(x) = 6x^2 - 13x + 6$. If we consider $a_0 = 0, b_0 = 1$ in the false-position method, then it should print the following table.

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\lvert f(x_i^r)\rvert$
0	0.000000	1.000000	0.857143		0.285714	0.734694
1	0.000000	0.857143	0.763636	0.122449	0.145455	0.428430
2	0.000000	0.763636	0.712743	0.071405	0.069114	0.217643
3	0.000000	0.712743	0.687794	0.036274	0.031691	0.102959
4	0.000000	0.687794	0.676191	0.017160	0.014286	0.047076
5	0.000000	0.676191	0.670927	0.007846	0.006390	0.021191
6	0.000000	0.670927	0.668565	0.003532	0.002848	0.009472
7	0.000000	0.668565	0.667512	0.001579	0.001267	0.004220
8	0.000000	0.667512	0.667042	0.000703	0.000564	0.001878
9	0.000000	0.667042	0.666834	0.000313	0.000251	0.000835
10	0.000000	0.666834	0.666741	0.000139	0.000111	0.000371
11	0.000000	0.666741	0.666700	0.000062	0.000050	0.000165
12	0.000000	0.666700	0.666681	0.000027	0.000022	0.000073
13	0.000000	0.666681	0.666673	0.000012	0.000010	0.000033
14	0.000000	0.666673	0.666670	0.000005	0.000004	0.000014

The root is 0.666656. However, if we consider $a_0 = 1, b_0 = 3$ in the false-position method, then it takes more steps to converge.

Regula-Falsi Animation

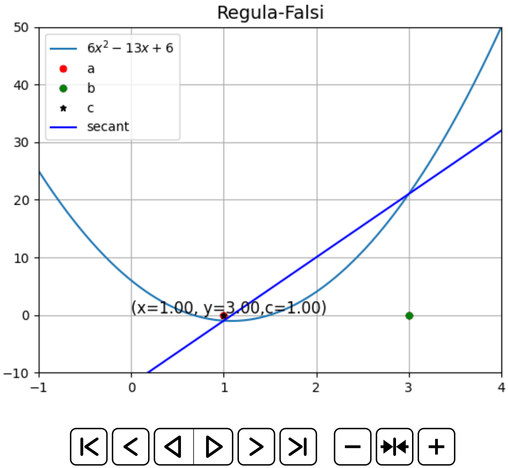

Figure 12: Regula-Falsi

Regula-Falsi Method

Example 7 (continued..)

Iteration (i)	a_i	b_i	x_i^r	ϵ_a	ϵ_t	$\|f(x_i^r)\|$
0	1.000000	3.000000	1.090909		0.272727	1.041322
1	1.090909	3.000000	1.181102	0.076364	0.212598	0.984314
2	1.181102	3.000000	1.262541	0.064504	0.158306	0.848975
3	1.262541	3.000000	1.330052	0.050759	0.113298	0.676445
4	1.330052	3.000000	1.382165	0.037704	0.078556	0.505863
5	1.382165	3.000000	1.420220	0.026795	0.053186	0.360710
6	1.420220	3.000000	1.446897	0.018437	0.035402	0.248594
7	1.446897	3.000000	1.465068	0.012402	0.023288	0.167340
8	1.465068	3.000000	1.477202	0.008215	0.015199	0.110871
9	1.477202	3.000000	1.485200	0.005385	0.009867	0.072687
10	1.485200	3.000000	1.490425	0.003506	0.006384	0.047326
11	1.490425	3.000000	1.493819	0.002272	0.004121	0.030675
12	1.493819	3.000000	1.496016	0.001469	0.002656	0.019825
13	1.496016	3.000000	1.497434	0.000947	0.001710	0.012788
14	1.497434	3.000000	1.498349	0.000610	0.001101	0.008239
15	1.498349	3.000000	1.498938	0.000393	0.000708	0.005304
16	1.498938	3.000000	1.499317	0.000253	0.000455	0.003413
17	1.499317	3.000000	1.499561	0.000163	0.000293	0.002195
18	1.499561	3.000000	1.499718	0.000105	0.000188	0.001412
19	1.499718	3.000000	1.499818	0.000067	0.000121	0.000908
20	1.499818	3.000000	1.499883	0.000043	0.000078	0.000584
21	1.499883	3.000000	1.499925	0.000028	0.000050	0.000375
22	1.499925	3.000000	1.499952	0.000018	0.000032	0.000241
23	1.499952	3.000000	1.499969	0.000011	0.000021	0.000155

The root is 1.499980. Bisection method found this root in 2 steps.

Regula-Falsi Animation

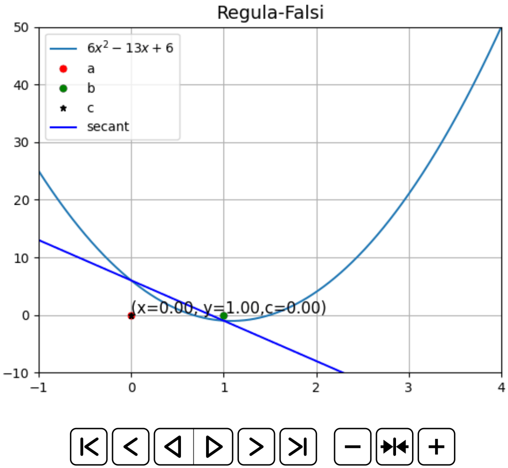

Figure 13: Regula-Falsi

Regula-Falsi Method

From the above two examples, you can observe that either a_i or b_i is not changing at all. Therefore, the false-position method has an onesidedness. The false-position method can also be written as

$$x = b - f(b)\frac{a-b}{f(a)-f(b)} = b - f(b)\frac{b-a}{f(b)-f(a)} \tag{1}$$

$$= a - f(a)\frac{b-a}{f(b)-f(a)} = a - f(a)\frac{a-b}{f(a)-f(b)} \tag{2}$$

$$= \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{bf(a)-af(b)}{f(a)-f(b)} \tag{3}$$

Regula-Falsi Method

Let $x_0 = a, x_1 = b$, then by the false-position method can be written as

$$x = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

If $f(x_0)f(x) < 0$, then set $x_2 = x$ and then

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

If $f(x_0)f(x_2) < 0$, then set our new interval as $[a', b'] = [x_0, x] = [x_1, x_2]$. Now, x_3 is computed as

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

After n iterations we obtain the following recurrence relations

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{4}$$

Regula-Falsi Method: Convergence

Regula-Falsi Method

Theorem 8

Let $f \in C^2[a,b]$ be such that $f(a)f(b) < 0$ and $\left|\frac{f''(x)}{2f'(x)}\right| < M, 0 \leq M \leq 1$. Suppose x_r is the root of f such that $f(x_r) = 0, f'(x_r) \neq 0$. Then for $x_0 = a$ and $x_1 = b$ sufficiently close to x_r, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the the false-position method as given in the recurrence relation (4) converges to x_r with the order of convergence as the golden ratio.

Regula-Falsi Method

Proof:

From the equation (4), we obtain that

$$\begin{aligned} x_{n+1} - x_r &= x_n - x_r - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \\ &= x_n - x_r - f(x_n)\frac{1}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} \end{aligned}$$

Regula-Falsi Method

Proof (continued using Interpolation Theorems):
Using Newton's divided difference formula

$$f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

and $f(x_r) = 0$, we obtain that

$$\begin{aligned} x_{n+1} - x_r &= x_n - x_r - \frac{f(x_n) - f(x_r)}{x_n - x_r} \frac{x_n - x_r}{f[x_{n-1}, x_n]} \\ &= \frac{(x_n - x_r)}{f[x_{n-1}, x_n]} \left[f[x_{n-1}, x_n] - \frac{f(x_r) - f(x_n)}{x_r - x_n} \right] \\ &= \frac{(x_n - x_r)(x_{n-1} - x_r)}{f[x_{n-1}, x_n]} \left[\frac{f[x_{n-1}, x_n] - f[x_n, x_r]}{x_{n-1} - x_r} \right] \end{aligned}$$

Regula-Falsi Method

Proof (continued):
Once again using the Newton's divided difference formula,

$$f[x_{n-1}, x_n, x_r] = \frac{f[x_{n-1}, x_n] - f[x_n, x_r]}{x_{n-1} - x_r}$$

and hence we obtain that

$$x_{n+1} - x_r = \frac{(x_n - x_r)(x_{n-1} - x_r)}{f[x_{n-1}, x_n]} f[x_{n-1}, x_n, x_r]$$

By the relations between the divided difference and derivatives theorem, there exists $\xi_n^1, \xi_n^2 \in (a, b)$ such that

$$f[x_{n-1}, x_n] = f'(\xi_n^1)$$
$$f[x_{n-1}, x_n, x_r] = \frac{1}{2!} f''(\xi_n^2)$$

Regula-Falsi Method

Proof (continued):
Hence, we obtain that

$$x_{n+1} - x_r = (x_n - x_r)(x_{n-1} - x_r)\frac{f''(\xi_n^2)}{2f'(\xi_n^1)}$$

$$|x_{n+1} - x_r| = |x_n - x_r||x_{n-1} - x_r|\left|\frac{f''(\xi_n^2)}{2f'(\xi_n^1)}\right|$$

Since

$$\left|\frac{f''(x)}{2f'(x)}\right| < M < 1, \forall x \in [a, b]$$

$$|x_{n+1} - x_r| \leq M_1|x_n - x_r||x_{n-1} - x_r|(Prove!)$$

Regula-Falsi Method

Suppose α is the order of convergence, then

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_r|}{|x_n - x_r|^\alpha} = C \implies |x_{n+1} - x_r| = K_1 |x_n - x_r|^\alpha$$

$$\implies |x_{n-1} - x_r| = K_2 |x_n - x_r|^{\frac{1}{\alpha}}$$

$$|x_{n+1} - x_r| = M_1 |x_n - x_r||x_{n-1} - x_r|$$

$$\implies K_1 |x_n - x_r|^\alpha = M_1 |x_n - x_r| K_2 |x_n - x_r|^{\frac{1}{\alpha}}$$

$$\implies |x_n - x_r|^\alpha = K_3 |x_n - x_r|^{1+\frac{1}{\alpha}}$$

Regula-Falsi Method

This implies that

$$\frac{1}{\alpha} + 1 = \alpha$$

Solving for α, we obtain that

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Hence the proof.

Regula-Falsi Method

Since the order of convergence is $\alpha > 1$, it is superlinear convergence. If we observe the pattern of this, for example,

$$
\begin{aligned}
|x_2 - x_r| &\leq M|x_1 - x_r||x_0 - x_r| \\
|x_3 - x_r| &\leq M|x_2 - x_r||x_1 - x_r| \leq M^2|x_1 - x_r|^2|x_0 - x_r| \\
|x_4 - x_r| &\leq M|x_3 - x_r||x_2 - x_r| \leq M^4|x_1 - x_r|^3|x_0 - x_r|^2 \\
|x_5 - x_r| &\leq M|x_4 - x_r||x_3 - x_r| \leq M^7|x_1 - x_r|^5|x_0 - x_r|^3 \\
|x_6 - x_r| &\leq M|x_5 - x_r||x_4 - x_r| \leq M^{12}|x_1 - x_r|^8|x_0 - x_r|^5 \\
|x_7 - x_r| &\leq M|x_6 - x_r||x_5 - x_r| \leq M^{20}|x_1 - x_r|^{13}|x_0 - x_r|^8 \\
|x_8 - x_r| &\leq M|x_7 - x_r||x_6 - x_r| \leq M^{33}|x_1 - x_r|^{21}|x_0 - x_r|^{13}
\end{aligned}
$$

Regula-Falsi Method

If we observe this pattern, we can see that

$$|x_{n+1} - x_r| \leq M^{\phi_{n+1}-1}|x_1 - x_r|^{\phi_n}|x_0 - x_r|^{\phi_{n-1}}$$

where ϕ_n denotes the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$. Since $M < 1$ and x_1, x_0 are sufficiently close to x_r, this sequence converges. In other words, if we consider the following, $y_n = M|x_n - x_r|$ and $z_n = \ln y_n$, we get

$$z_{n+1} = z_n + z_{n-1}$$

Solving this we obtain $\phi = 1.618$

Modified Regula-Falsi Method: Convergence

Modified Regula-Falsi Method

- The major weakness of the false position method is that its onesidedness.
- Therefore, one of the bracketing points will be always fixed. This leads to poor convergence.
- In order to overcome this, **modified false-position method** is used, where when the same endpoint is fixed twice, it uses $f(b_k)/2$ or $f(a_k)/2$ to accelerate the convergence.

Thanks

Doubts and Suggestions

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MA633L-Numerical Analysis

Lecture 16 : Solution of Nonlinear Equations: Bracketing Methods

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February 13, 2025