MA633L-Numerical Analysis

Lecture 16 : Solution of Nonlinear Equations: Open Methods

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Nonlinear Equations: Open Methods

Recap: Bracketing Methods

- For bracketing methods, the root is located within an interval.
- Iteratively applying the bracketing methods, we estimate a closer values to the true value of the root.
- These methods converge because they move closer to the truth.
- However, there is a disadvantage that, we have to find the two guesses one for a_0 and one for b_0 which brackets the roots.
- A wrong guess of either a_0 or b_0 will go vain.

Introduction: Open Methods

- In contrast, the open methods require a single starting value or two starting values that do not necessarily bracket the root.
- Open methods converge much faster than bracketing methods.
- However, the disadvantage of open methods is that, it can diverge or move away from the true root.

Fixed Point Iterations

Fixed Point

Definition 1 (Fixed Point)

A fixed point or invariant point of a function $f : X \to X$ is an element $x \in X$ that mapped to itself. That is, $f(x) = x$.

Example 2

Find the fixed points of the map $f : [0, 2] \to [0, 2]$ defined by $f(x) = x^3 - 3x^2 + 3x$ Is f a self map? (Prove!)

$$f(x) = x \implies x^3 - 3x^2 + 2x = 0$$

$$\implies x(x^2 - 3x + 2) = 0$$

$$\implies x(x-1)(x-2) = 0 \implies x = 0, 1, 2$$

Fixed Point Example

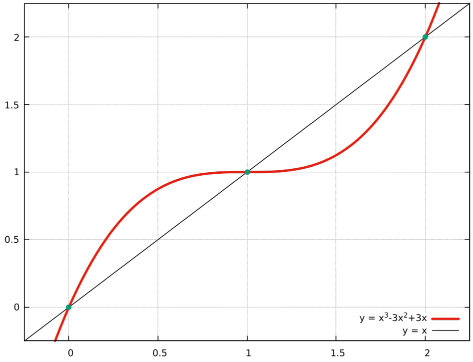

Figure 1: $f(x) = x$ curve

Fixed Point Example

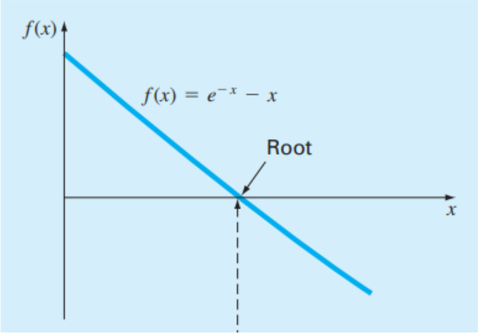

Figure 2: $e^{-x} - x$ curve

Fixed Point Example

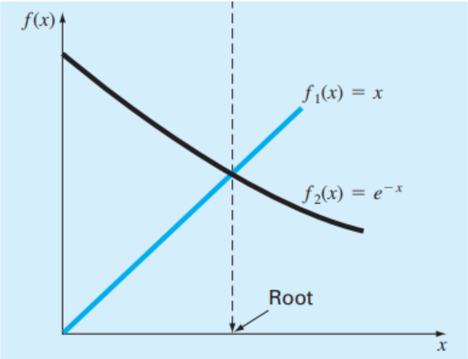

Figure 3: $y = e^{-x}$ and $y = x$ curves

Fixed Point Iterations

Theorem 3

Every $f \in C([a,b],[a,b])$ has a fixed point.

Proof:

If $f(a) = a$ or $f(b) = b$, then we are done. Suppose $f(a) > a$ and $f(b) < b$. Now define $g(x) = f(x) - x$. Since f is continuous, g is continuous. Further $g(a) > 0$ and $g(b) < 0$. Therefore, by intermediate value theorem, there is an $x \in (a,b)$ such that $g(x) = 0 \implies f(x) = x$.

Fixed Point Iterations

Theorem 4
Every $f \in C^1([a,b],[a,b])$ with $|f'(x)| < 1, \forall x \in (a,b)$ has a unique fixed point in $[a,b]$.

Proof:
Since f is differentiable, it is continuous and hence by previous theorem f has a fixed point. Now, suppose f has a two fixed points say $x_1, x_2 \in [a,b]$. Since $x_1 \neq x_2$, by mean value theorem, we have

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(\xi), \quad \xi \in (a,b)$$

$$\implies |x_1 - x_2| = |f(x_1) - f(x_2)| = |f'(\xi)||x_1 - x_2| < |x_1 - x_2|$$

which is a contradiction. Therefore, $x_1 = x_2$ and hence the proof.

Fixed Point

- For a given equation $g(x) = 0$, rearrange it to the following form $f(x) = x$ which can be simply obtained either by taking $f(x) = g(x) + x$ or doing a few manipulation.
- Examples:

$$ax^2 + bx + c = 0$$

can be rearranged as

$$x = \frac{-c - ax^2}{b}, b \neq 0 \text{ or } x = -\frac{-c}{ax + b} \text{ or } x = -\frac{c + bx}{ax}$$

whereas as

$$x \sin(x) = 0$$

can be rearranged as

$$x + x \sin(x) = x$$

Fixed Point Iterations

- Now starting with a guess x_0, we obtain that

$$x_1 = f(x_0)$$

- Then

$$x_2 = f(x_1)$$

Repeatedly applying, we obtain that

$$x_{n+1} = f(x_n)$$

- Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence generated by the fixed point iteration and converges to x_r, then

$$x_r = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_r)$$

Fixed Point Iterations

Example 5

Use simple fixed point iteration to locate the root of $f(x) = e^{-x} - x$.

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert g(x_i)\rvert$
0	0.000000		1.000000	1.000000
1	1.000000	1.000000	0.367879	0.632121
2	0.367879	1.718282	0.692201	0.324321
3	0.692201	0.468536	0.500474	0.191727
4	0.500474	0.383091	0.606244	0.105770
5	0.606244	0.174468	0.545396	0.060848
6	0.545396	0.111566	0.579612	0.034217
7	0.579612	0.059034	0.560115	0.019497
8	0.560115	0.034809	0.571143	0.011028
9	0.571143	0.019308	0.564879	0.006264
10	0.564879	0.011089	0.568429	0.003549
11	0.568429	0.006244	0.566415	0.002014
12	0.566415	0.003556	0.567557	0.001142
⋮	⋮	⋮	⋮	⋮
21	0.567148	0.000022	0.567141	0.000007
22	0.567141	0.000012	0.567145	0.000004
23	0.567145	0.000007	0.567142	0.000002

The root is 0.567145.

Fixed Point Animation

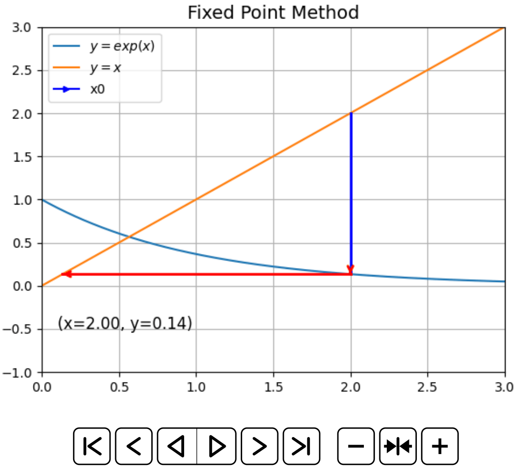

Figure 4: Fixed Point

Fixed Point Iterations: Convergence

Fixed Point Iterations

Theorem 6 (Fixed Point Theorem)
Let $f \in C^1([a,b],[a,b])$ with $|f'(x)| \leq k < 1, \forall x \in (a,b)$. Then for any $x_0 \in [a,b]$, the sequence defined by $x_{n+1} = f(x_n)$ converges to the unique fixed point $x_r \in [a,b]$.

Proof:
By above theorem, there is a unique fixed point $x_r \in [a,b]$. Now

$$|x_{n+1} - x_r| = |f(x_n) - f(x_r)| = |f'(\xi_n)||x_n - x_r| \leq k|x_n - x_r|$$

Applying repeatedly, we obtain that

$$|x_{n+1} - x_r| \leq k|x_n - x_r| \leq k|x_{n-1} - x_r| \leq \cdots \leq k^{n+1}|x_0 - x_r|$$

Since $0 < k < 1$, $k^{n+1} \to 0$ and hence

$$\lim_{n\to\infty} |x_{n+1} - x_r| \to 0.$$

Hence the proof.

Fixed Point Iterations

Theorem 7

The error bound for the fixed point approximation is given by

$$|x_{n+1} - x_r| \leq k^{n+1} \max\{x_0 - a, b - x_0\}$$

and

$$|x_n - x_r| \leq \frac{k^n}{1-k} |x_1 - x_0|$$

Proof:

By above theorem, we have observed that

$$|x_{n+1} - x_r| \leq k^{n+1} |x_0 - x_r| \leq k^{n+1} \max\{x_0 - a, b - x_0\}$$

Fixed Point Iterations

Also,

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(\xi_n)||x_n - x_{n-1}| \leq k|x_n - x_{n-1}|$$

$$|x_{n+1} - x_n| \leq k^n |x_1 - x_0|$$

Now, $m > n \geq 1$,

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\leq k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \cdots k^n|x_1 - x_0| \\ &= k^n|x_1 - x_0|(1 + k + k^2 + \cdots + k^{m-n-1}) \end{aligned}$$

Fixed Point Iterations

Since $\lim_{m\to\infty} x_m = x_r$, we have

$$
\begin{aligned}
|x_r - x_n| &= \lim_{m\to\infty} |x_m - x_n| \\
&\le \lim_{m\to\infty} k^n |x_1 - x_0| \sum_{i=0}^{m-n-1} k^i \\
&\le k^n |x_1 - x_0| \sum_{i=0}^{\infty} k^i \\
&= \frac{k^n}{1-k} |x_1 - x_0|
\end{aligned}
$$

Fixed Point Iterations

Remarks

- This method is also called as one-point iteration or successive substitution method.
- These inequalities relates the rate at which the sequence converges to the bound k on the first derivative.
- The rate of convergence depends on the factor k^n.
- Smaller the value of k, faster the convergence.
- However due to the second inequality if k is close to 1, the convergence is very slow.

Fixed Point Iterations

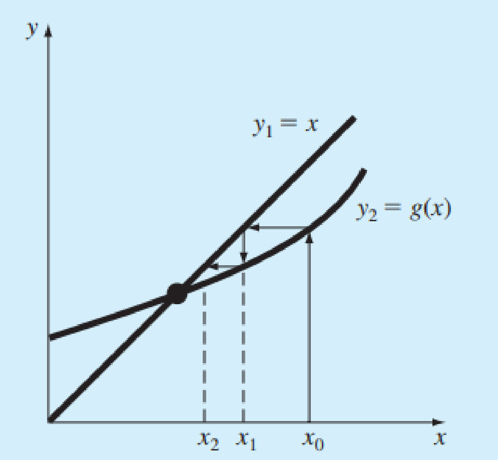

Figure 5: Convergence of Fixed Point

Fixed Point Iterations

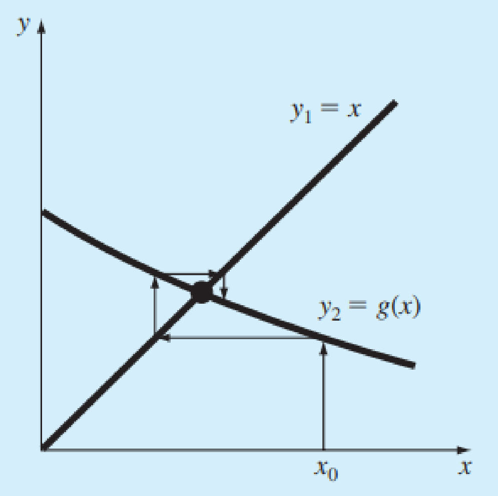

Figure 6: Convergence of Fixed Point

Fixed Point Iterations

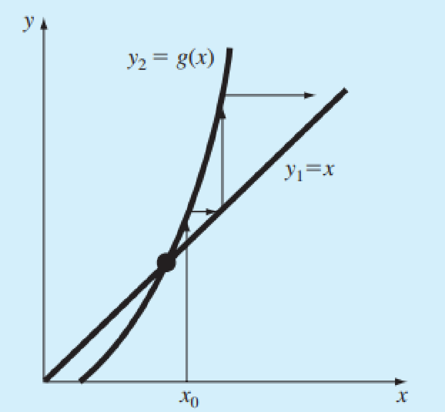

Figure 7: Divergence of Fixed Point

Fixed Point Iterations

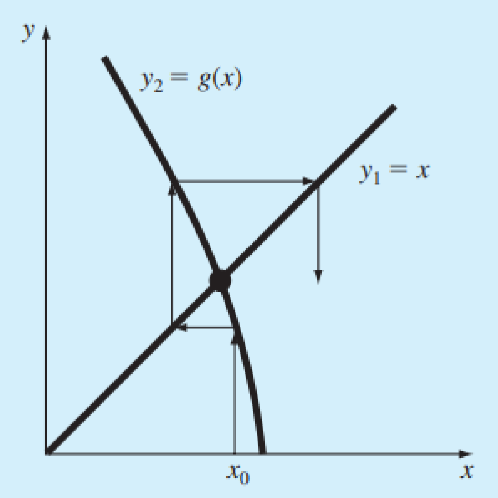

Figure 8: Divergence of Fixed Point

Fixed Point Iterations

Theorem 8
Let $f \in C^1([a, b], [a, b])$ with $|f'(x)| \leq k < 1, \forall x \in (a, b)$. Then the order of convergence is 1.

By fixed point theorem, we have

$$|x_{n+1} - x_r| = |f(x_n) - f(x_r)| = |f'(\xi_n)||x_n - x_r| \leq k|x_n - x_r|$$

$$\frac{|x_{n+1} - x_r|}{|x_n - x_r|} \leq k$$

Hence the proof.

Newton-Raphson Method

Newton-Raphson Method

- Newton-Raphson method is widely used in many applications.
- The key idea behind the Newton-Raphson method is that for the initial guess at the root x_n, a tangent can be extended from the point $(x_n, f(x_n))$.
- The point where this tangent crosses the x-axis usually represents the improved estimate of the root.
- The Newton-Raphson can be derived from Taylor theorem.

Newton-Raphson Method

Suppose $f \in C^2[a,b]$. Let $x_n \in [a,b]$ be an approximation to x_r such that $f'(x_n) \neq 0$ and $|x_r - x_n| < \delta$.

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{f''(\xi)}{2!}(x - x_n)^2$$

Now if we set $x = x_{n+1}$ and for a large n, $f(x_{n+1}) = f(x_r) = 0$, then

$$\begin{aligned} f(x_{n+1}) &= f(x_n) + (x_{n+1} - x_n)f'(x_n) + \frac{f''(\xi)}{2!}(x_{n+1} - x_n)^2 \\ 0 &= f(x_n) + (x_{n+1} - x_n)f'(x_n) + O(x_{n+1} - x_n)^2 \end{aligned}$$

Newton-Raphson Method

Since δ is small, neglecting and rearranging, we obtain that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

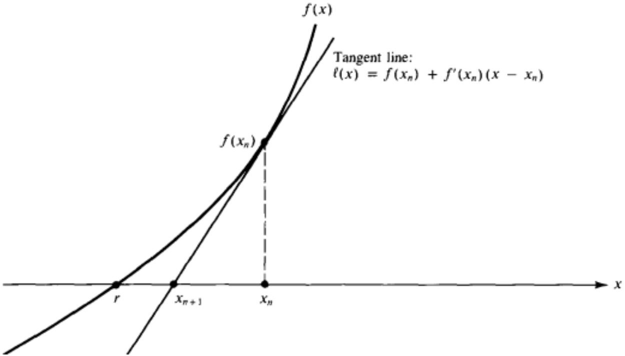

Figure 9: Geometrical Interpretation of Newton-Raphson Method

Newton-Raphson Method

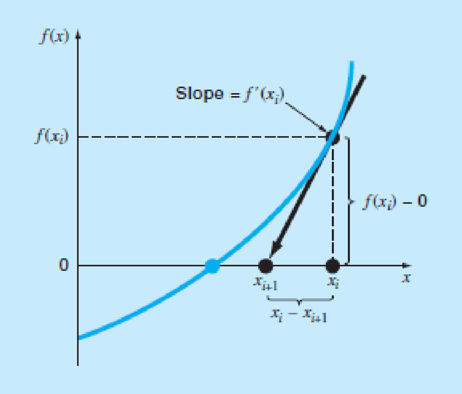

Figure 10: Geometrical Interpretation of Newton-Raphson Method

Newton-Raphson Method

For an initial guess x_0, the Newton-Raphson method generates a sequence $\{x_n\}_{n=0}^{\infty}$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

This equation is called Newton-Raphson formula.

Newton-Raphson Method

Example 9

Use simple Newton-Raphson method to locate the root of $f(x) = e^{-x} - x$ with $x_0 = 0$.

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert f'(x_i)\rvert$
0	0.000000		1.000000	2.000000
1	0.500000	1.000000	0.106531	1.606531
2	0.566311	0.117093	0.001305	1.567616
3	0.567143	0.001467	0.000000	1.567143
4	0.567143	0.000000	0.000000	1.567143

The root is 0.567145. Compared to the fixed point iteration, you can see that, this converges much faster.

Newton-Raphson Animation

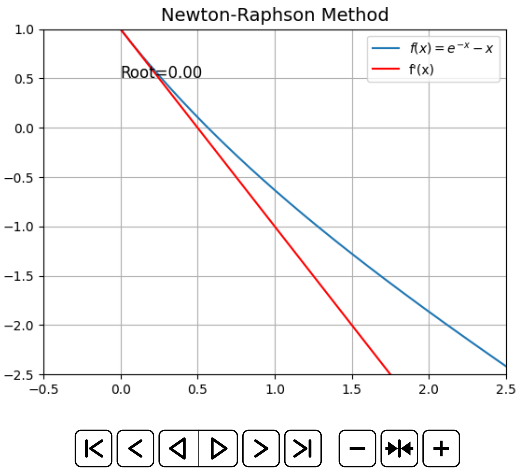

Figure 11: Fixed Point

Newton-Raphson Method

Example 10

Use simple Newton-Raphson method to locate the root of $f(x) = \cos(x) - x$ with $x_0 = \pi/4$.

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert f'(x_i)\rvert$
0	0.785398		0.078291	1.707107
1	0.739536	0.062015	0.000755	1.673945
2	0.739085	0.000610	0.000000	1.673612
3	0.739085	0.000000	0.000000	1.673612

The root is 0.739085.

Newton-Raphson Animation

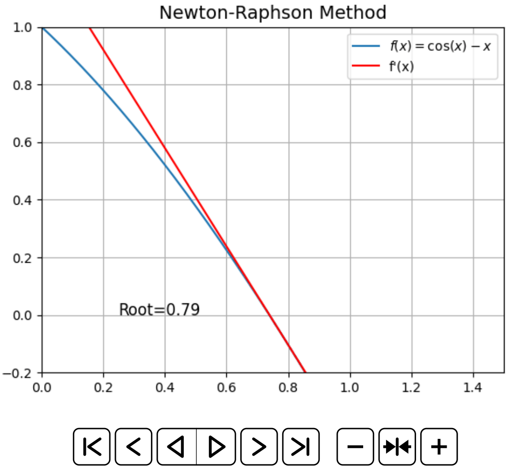

Figure 12: Fixed Point

Newton-Raphson Method

Example 11

Use simple Newton-Raphson method to locate the root of $f(x) = e^x - 1.5 - \tan^{-1}(x)$, with $x_0 = -7$.

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i) \rvert$	$\lvert f'(x_i) \rvert$
0	-7.000000		0.070189	0.019088
1	-10.677096	0.344391	0.022567	0.008673
2	-13.279167	0.195951	0.004366	0.005637
3	-14.053656	0.055109	0.000239	0.005037
4	-14.101110	0.003365	0.000001	0.005003
5	-14.101270	0.000011	0.000000	0.005003
6	-14.101270	0.000000	0.000000	0.005003

The root is -14.101270.

Newton-Raphson Animation

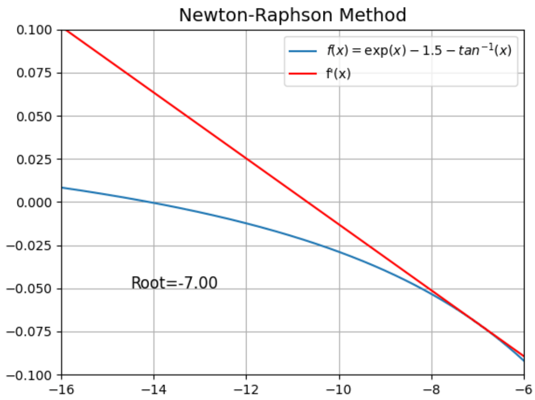

Figure 13: Fixed Point

Newton-Raphson Method

Example 12

Use simple Newton-Raphson method to locate the root of $f(x) = x^2 + 1$, with $x_0 = 0.5$.

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert f'(x_i)\rvert$
0	0.500000		1.250000	1.000000
1	-0.750000	1.666667	1.562500	1.500000
2	0.291667	3.571429	1.085069	0.583333
3	-1.568452	1.185958	3.460043	3.136905
4	-0.465441	2.369823	1.216635	0.930881
5	0.841531	1.553088	1.708174	1.683061
6	-0.173390	5.853393	1.030064	0.346780
7	2.796975	1.061992	8.823069	5.593950
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$\ldots$	$\ldots$	$\ldots$	$\ldots$

This will never converge. No matter whatever real starting point is selected.

Newton-Raphson Animation

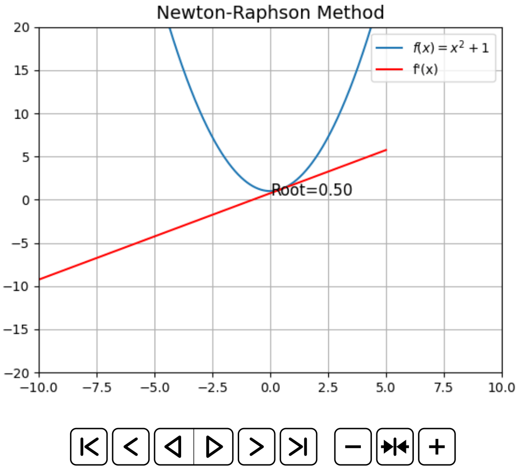

Figure 14: Fixed Point

Newton-Raphson Method

Example 13

Use Newton-Raphson method to locate the root of $f(x) = x^{10} - 1$, with $x_0 = 0.5$.

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert f'(x_i)\rvert$
0	0.500000		0.999023	0.019531
1	51.650000	0.990319	135114904483913696.000000	26159710451871000.000000
2	46.485000	0.111111	47111654129711536.000000	10134807815362276.000000
3	41.836500	0.111111	16426818072478544.000000	3926432199748675.000000
⋮	⋮	⋮	⋮	⋮
10	20.010268	0.111111	10292695105054.697266	5143706707446.162109
11	18.009241	0.111111	3588840873655.112793	1992777367871.565186
12	16.208317	0.111111	1251351437592.922363	772042782329.150146
⋮	⋮	⋮	⋮	⋮
43	1.000000	0.000000	0.000000	10.000000

It converges after 43 steps, and this convergence rate is very slow.

Newton-Raphson Animation

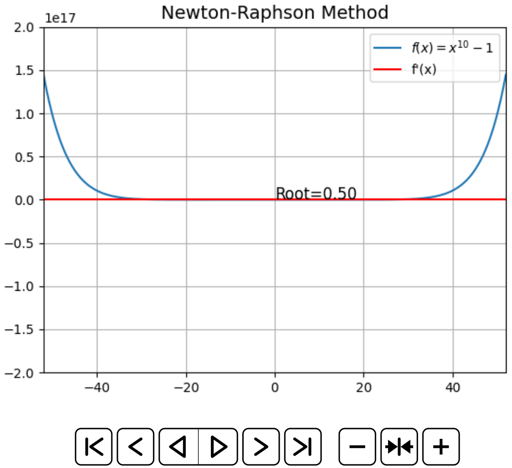

Figure 15: Fixed Point

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in

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Lecture 16 : Solution of Nonlinear Equations: Open Methods

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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भारतीय प्रौद्योगिकी संस्थान तिरुपति
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