MA633L-Numerical Analysis

Lecture 19 : Solution of Nonlinear Equations: Open Methods - Secant Method

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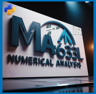

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Nonlinear Equations: Open Methods

Recap: Bracketing Methods

- For bracketing methods, the root is located within an interval.
- Iteratively applying the bracketing methods, we estimate a closer values to the true value of the root.
- These methods converge because they move closer to the truth.
- However, there is a disadvantage that, we have to find the two guesses one for a_0 and one for b_0 which brackets the roots.
- A wrong guess of either a_0 or b_0 will go vain.

Introduction: Open Methods

- In contrast, the open methods require a single starting value or two starting value that do not necessarily bracket the root.
- Open methods converge much faster that bracketing methods.
- However, the disadvantage of open methods is that, it can diver or move away from the true root.

Newton-Raphson Method

Newton-Raphson Method From Fixed Point

The easiest way to construct a fixed point iteration associated with a root-finding problem $f(x) = 0$ is to subtract $f(x)$ from x. So, let us consider

$$x_{n+1} = g(x_n)$$

where g is of the form

$$g(x) = x - \phi(x)f(x)$$

where ϕ is a differentiable function. For the iterative scheme to be quadratically convergence, we need to have $g'(x_r) = 0$ when $f(x_r) = 0$

$$g'(x) = 1 - \phi'(x)f(x) - f'(x)\phi(x) \implies \phi(x_r) = 1/f'(x_r)$$

So, choose,

$$\phi(x) = 1/f'(x)$$

Secant Method

Secant Method

- As we have seen earlier, Newton-Raphson method requires calculation of derivatives always.
- Instead of this, Secant method uses an equivalent method to compute the roots as a general-purpose procedure, however the convergence is as fast as Newton-Raphson method.
- However, Secant method requires two starting values.

Secant Method

As per the Newton-Raphson method we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

By the definition of derivative we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Secant Method

In particular, let us choose, $x = x_n$ and $h = x_{n-1} - x_n$, we have

$$f'(x) \approx \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

Therefore, the Newton-Raphson method becomes

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Secant Method

- The name Secant is used because on the right hand side instead of the tangent $f'(x)$ we have used the slope of the secant line to the graph of f.
- For the same reason, Newton-Raphson method is called as tangent method.
- From the equation, it is clear that we require two starting values say x_0, x_1.
- Then the sequence can generate the rest.
- Once again if $f(x_n) - f(x_{n-1})$ are nearly zero, either loss of significant digits or overflow error can occur.

Secant Method : Convergence Analysis

From the comparison of False-position method and Secant method you can observe that both are almost same and hence the order of convergence of Secant method is also the golden ratio.

Secant Method

Example 1

Use Secant method to locate the root of $f(x) = e^x - 1.5 - \tan^{-1}(x)$, with $x_0 = -7$ and $x_1 = -10.0$.

Iteration (i)	x_{i-1}	x_i	x_{i+1}	$\lvert f(x_{i+1}) \rvert$
0	-7.000000	-10.000000	-12.090831	0.011718
1	-10.000000	-12.090831	-13.522760	0.003017
2	-12.090831	-13.522760	-14.019379	0.000412
3	-13.522760	-14.019379	-14.097931	0.000017
4	-14.019379	-14.097931	-14.101250	0.000000

The root is -14.101250. N-R took, 6 steps.

Secant Method

Example 2

Use Secant method to locate the root of $f(x) = \cos(x) - x$ with $x_0 = 0$ and $x_1 = \pi/4$.

Iteration (i)	x_{i-1}	x_i	x_{i+1}	$\lvert f(x_{i+1}) \rvert$
0	0.000000	0.785398	0.728373	0.017886
1	0.785398	0.728373	0.738978	0.000180
2	0.728373	0.738978	0.739085	0.000000

The root is 0.739085. N-R Method took, 3 steps.

Secant Method

Example 3

Use Secant method to locate the root of $f(x) = x^{10} - 1$ with $x_0 = 0$, $x_1 = 0.78$.

Iteration (i)	x_{i-1}	x_i	x_{i+1}	$\lvert f(x_{i+1})\rvert$
0	1.200000	2.000000	1.195919	4.984373
1	2.000000	1.195919	1.191982	4.790263
2	1.195919	1.191982	1.094827	1.474313
3	1.191982	1.094827	1.051630	0.654365
4	1.094827	1.051630	1.017157	0.185442
5	1.051630	1.017157	1.003524	0.035806
6	1.017157	1.003524	1.000262	0.002623
7	1.003524	1.000262	1.000004	0.000041
8	1.000262	1.000004	1.000000	0.000000

The root is 1.0. N-R took 43 steps.

Multiple Roots

Multiple Roots

One of the disadvantage of Newton-Raphson and Secant method is that we must have $f'(x_r) \neq 0$ and $f(x_r) \neq 0$. To overcome this difficulty, let us employ another way.

Definition 4

Multiplicity: A root x_r of $f(x) = 0$ is a root of multiplicity m of f if for $x \neq x_r$, we can write

$$f(x) = (x - x_r)^m Q(x)$$

where

$$\lim_{x \to x_r} Q(x) \neq 0$$

Theorem 5

The function $f \in C^1[a, b]$ has a simple root at x_r in (a, b) if and only if $f(x_r) = 0$ and $f'(x_r) \neq 0$.

Multiple Roots

Theorem 6
The function $f \in C^m[a,b]$ has a root of multiplicity m at x_r in (a,b) if and only if

$$f(x_r) = f'(x_r) = f''(x_r) = f^{(m-1)}(x_r) = 0 \quad \text{but} \quad f^{(m)}(x_r) \neq 0$$

If a root is simple root, then Newton-Raphson method can be applied. Now, let us define a new function

$$\mu(x) = \frac{f(x)}{f'(x)}$$

If x_r is a root of f of multiplicity m with $f(x) = (x - x_r)^m Q(x)$, then

$$f'(x) = m(x - x_r)^{m-1} Q(x) + (x - x_r)^m Q'(x)$$

Multiple Roots

$$\mu(x) = \frac{(x - x_r)^m Q(x)}{m(x - x_r)^{m-1} Q(x) + (x - x_r)^m Q'(x)}$$
$$= (x - x_r) \frac{Q(x)}{mQ(x) + (x - x_r)Q'(x)}$$

At $x = x_r$, we have $\mu(x_r) = 0$. Now,

$$\mu'(x) = \frac{Q(x)}{mQ(x) + (x - x_r)Q'(x)} + (x - x_r) \left[\frac{Q(x)}{mQ(x) + (x - x_r)Q'(x)} \right]'$$

Multiple Roots

At $x = x_r$, we have

$$\mu'(x_r) = \frac{1}{m}$$

If we apply the Newton-Raphson method for the function $\mu(x)$, we obtain that

$$x_{n+1} = x_n - \frac{\mu(x_n)}{\mu'(x_n)} = x - \frac{f(x_n)}{f'(x_n)} \frac{f'(x_n)^2}{f'(x_n)^2 - f(x_n)f''(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}$$

The above iteration provides the root of the function f under certain continuity and differentiability conditions and it converges quadratically for the root of multiplicity m of f. Here, we need more work on f with computation of f'' also.

Comparison

Comparison of Methods

Method	Initial Guess	Formula
Bisection	a_0, b_0 with $f(a_0)f(b_0) < 0$	$x_n^r = \frac{a_n+b_n}{2}$ change $a_{n+1} = x_n^r$ or $b_{n+1} = x_n^r$
False Position	a_0, b_0 with $f(a_0)f(b_0) < 0$	$x_n^r = b_n - f(b_n)\frac{a_n-b_n}{f(a_n)-f(b_n)}$ change $a_{n+1} = x_n^r$ or $b_{n+1} = x_n^r$
Fixed Point	x_0, then rewrite $g(x) = 0$ as $f(x) = x$	$x_{n+1} = f(x_n)$
Newton-Raphson	x_0	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
Secant	x_0, x_1	$x_{n+1} = x_n - \frac{f(x_n)(x_{n-1}-x_n)}{f(x_{n-1})-f(x_n)}$ $x_{n+1} = x_n$ $x_n = x_{n-1}$
Modified Newton-Raphson for multiple roots	x_0	$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2-f(x_n)f''(x_n)}$

Comparison of Methods

Method	Merits	Demerits	Failures
Bisection	Always converge Linear Convergence	Slow Convergence	For complex roots or even multiple roots
False Position	Always converge, 1.618 Faster than Bisection convergence order 1.618	Slow Convergence	For complex roots or even multiple roots for significant curvature
Fixed Point	Simple to compute Linear Convergence	Converges when $\|f'(x)\| < 1$	Fails for $\|f'(x)\| > 1$
Newton-Raphson	Widely used, Converges faster if it does Quadratic convergence	Finding derivative is difficult, diverge or poor convergence	inflection point extrema tangent or slope zero
Secant	No derivatives faster if it does convergence order 1.618	Selecting initial guess is difficult, diverge or poor convergence	inflection point extrema tangent or slope zero
Modified Newton-Raphson for multiple roots	Useful for multiples root cases Quadratic convergence	Finding derivatives are difficult, inefficient for simple roots	Not useful for functions with simple roots

Complex Roots and Müller’s Method

Müller's Method

One problem with applying the Secant, False Position, or Newton's method to polynomials is the possibility of the polynomial having complex roots even when all the coefficients are real numbers. If the initial approximation is a real number, all subsequent approximations will also be real numbers. One way to overcome this difficulty is to begin with a complex initial approximation and do all the computations using complex arithmetic. An alternative approach has its basis in the following theorem.

Theorem 7

If $z = a + ib$ is a complex root of multiplicity m of the polynomial $P_n(x)$ with real coefficients, then $z = a - ib$ is also a root of multiplicity m of the polynomial $P_n(x)$, and

$$P_n(x) = (x^2 - 2ax + a^2 + b^2)^m Q(x)$$

Polynomials

Polynomials

Many physics, engineering and real life problems are getting benefits of polynomials, for example, third equation of motion

$$u^2 - v^2 = 2as$$

in finance to compute the interest

$$A = P\frac{i(1+i)^n}{(1+i)^n - 1}$$

in RNN, the Legendre polynomials

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Polynomials

In control systems, transfer functions for a robotic position system is given by

$$G(s) = \frac{s^3 + 9s^2 + 26s + 24}{s^4 + 15s^3 + 77s^2 + 153s + 90}$$

Finding the roots of a polynomial is another important task in engineering sciences. For example, when we solve an ordinary differential equations with constant coefficients, we will end up with a polynomial. In order to find the solution of this differential equation, we must solve this polynomial.

Quadratic/Cubic Equations

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

In general, the roots of the Cubic equation (1) is given by

$$x_k = \frac{1}{3a}\left(b + \mu^k P + \frac{\Delta_0}{\mu^k P}\right), k = 0, 1, 2$$

$$\mu = \frac{-1 + \sqrt{-3}}{2}, P = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

$$\Delta_0 = b^2 - 3ac, \Delta_1 = 2b^3 - 9abc + 27a^2 d$$

Cubic Equations

Example 8

Find the roots of the following equation $x^3 - 5x^2 + 8x - 4 = 0$

$$
\begin{aligned}
\mu &= \tfrac{-1+\sqrt{3}i}{2} \\
\Delta_0 &= 25 - 24 = 1 \\
\Delta_1 &= -250 + 360 - 108 = 2 \\
P &= \sqrt[3]{\tfrac{2 \pm \sqrt{2^2 - 4 * 1^3}}{2}} = 1 \\
x_0 &= \tfrac{1}{3}\left(-5 + 1 + \tfrac{1}{1}\right) = 1 \\
x_1 &= \tfrac{1}{3}\left(-5 + \tfrac{-1+\sqrt{3}i}{2} + \tfrac{2}{-1+\sqrt{3}i}\right) = 2 \\
x_2 &= \tfrac{1}{3}\left(-5 + \left(\tfrac{-1+\sqrt{3}i}{2}\right)^2 + \left(\tfrac{2}{-1+\sqrt{3}i}\right)^2\right) = 2
\end{aligned}
$$

Therefore, the roots are $1, 2, 2$.

Quartic Equation

The four roots of the general quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \tag{2}$$

is given by

$$x = -\frac{b}{4a} \pm S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}$$

$$S = \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3a}\left(Q + \frac{\Delta_0}{Q}\right)}, Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

$$p = \frac{8ac - 3b^2}{8a^2}, q = \frac{b^3 - 4abc + 8a^2 d}{8a^3}$$

$$\Delta_0 = c^2 - 3bd + 12ae$$

$$\Delta_1 = 2c^3 - 9bcd + 27b^2 e + 27ad^2 - 72ace$$

Fundamental Theorem of Algebra

A polynomial $P_n(x)$ of degree n has the form

$$P_n(x) = \sum_{i=0}^{n} a_i x^i \tag{3}$$

Theorem 9 (Fundamental Theorem of Algebra)
Every nonconstant polynomial has at least one zero in the complex field.

Remainder Theorem

Note that fundamental theorem does not guarantee the existence of real roots for even simple problems such as $x^2 + 1 = 0$ although all its coefficients are real.

Theorem 10 (Remainder Theorem)
If we divide a polynomial by $(x - x_r)$ then we obtain

$$P_n(x) = (x - x_r)Q_{n-1}(x) + P_n(x_r)$$

where $Q_{n-1}(x)$ is a polynomial of degree $n - 1$ and $P_n(x_r)$ is the remainder

Factor Theorem

Let us say a root of $P_n(x)$ is x_{r_1}, then $P_n(x) = (x - x_{r_1})Q_{n-1}(x)$. If $Q_{n-1}(x)$ is a nonconstant polynomial of degree ≥ 1, then by fundamental theorem $Q_{n-1}(x)$ has a root say x_{r_2}. Hence, we can write

$$P_n(x) = (x - x_{r_1})(x - x_{r_2})R_{n-2}(x)$$

Factor Theorem

Repeating this procedure we obtain the following theorem.

Theorem 11 (Factor Theorem)
If $P_n(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exists unique constants $x_{r_1}, x_{r_2}, \cdots, x_{r_k}$, possibly complex and unique positive integers $m_1, m_2, \cdots, m_k$ such that

$$P_n(x) = a_n(x - x_{r_1})^{m_1}(x - x_{r_2})^{m_2} \cdots (x - x_{r_k})^{m_k} \tag{4}$$

where

$$\sum_{i=0}^{k} m_i = n$$

Here m_i is the multiplicity of the root x_{r_i}.

Locate the Roots

We have identified that if the initial guesses are near the root, then most probably methods will converge to the root. However, how to localize the roots is a question. The following theorem helps to identify the upper bound for the roots.

Theorem 12 (Locate the Roots)

All roots of a polynomial $P_n(x)$ lie in a annulus in the complex plane whose inner radius r and outer radius R is given by

$$\frac{1}{r} = 1 + \frac{1}{|a_0|} \max_{0 \leq i \leq n} |a_i| \tag{5}$$

$$R = 1 + \frac{1}{|a_n|} \max_{0 \leq i \leq n} |a_i| \tag{6}$$

Locate the Roots

Example 13

Find the annulus that contains all the roots of the polynomial

$$P_4(x) = x^4 - 4x^3 + 7x^2 - 5x - 2$$

$$R = 1 + |a_4|^{-1} \max_{0 \le i \le 4} |a_i| = 8 \quad \text{and} \quad \frac{1}{r} = 1 + |a_0|^{-1} \max_{0 \le i \le 4} |a_i| = \frac{9}{2}$$

Therefore, all the roots of the polynomial $P_4(x)$ lies in the annulus $\frac{2}{9} < |z| < 8$

Horner's Method

Horner's Method

When we apply the Newton-Raphson method to find the roots of a polynomial $P_n(x)$, we obtain that

$$x_{n+1} = x_n - \frac{P_n(x)}{P_n'(x)} \tag{7}$$

Using theorem (12) we can identify the disk or annulus where the roots are located. However, to find the approximate roots of the polynomial $P_n(x)$, we need to compute both $P_n(x)$ and $P_n'(x)$ at prescribed values at each step.

Horner's Method

In order to compute the polynomial evaluation in computer, it is better to use the following method. A polynomial $P_n(x)$

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

can also be written as

$$P_n(x) = (\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots a_1)x + a_0$$

Horner's Method

Theorem 14 (Horner's Method)

Let

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

Define $b_n = a_n$ and

$$b_k = a_k + b_{k+1} x_0, \quad \text{for} \quad k = n-1, n-2, \cdots 1, 0,$$

then $P_n(x_0) = b_0$. Further,

$$P_n(x) = (x - x_0) Q_{n-1}(x) + b_0$$

where

$$Q_{n-1}(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$$

Horner’s Method

Horner’s method uses the nesting technique to compute the n-degree polynomial which requires only n multiplications and n additions. Also it has an advantage that when the Newton-Raphson method is implemented, both $P_n(x)$ and $P_n'(x)$ can be evaluated in the same manner as $P_n'(x_0) = Q_{n-1}(x_0)$. With the assistance of Horner’s method, Newton-Raphson iteration we will obtain a simple root x_{r_1} of the polynomial $P_n(x)$. In such case the polynomial can be written as

$$P_n(x) = (x - x_{r_1})Q_{n-1}(x)$$

Horner's Method

Upon obtaining the root of x_{r_1}, we need to obtain $Q_{n-1}(x)$, which is called **polynomial deflation** as the degree of $Q_{n-1}(x)$ is one less than the polynomial $P_n(x)$. The below algorithm produces the deflated polynomial. Note that if x_{r_1} is a simple root, then b_0 will become zero from this algorithm. If x_{r_1} is a root, then b_0 will survive. We can now employ the Newton-Raphson method again on $Q_{n-1}(x)$ and obtain that

$$Q_{n-1}(x) = (x - x_{r_2})R_{n-2}(x)$$

$$P_n(x) = (x - x_{r_1})(x - x_{r_2})R_{n-2}(x)$$

Thanks

Doubts and Suggestions

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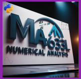

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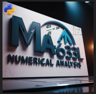

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