MA633L-Numerical Analysis

Lecture 2 : Preliminaries and Asymptotic Notations

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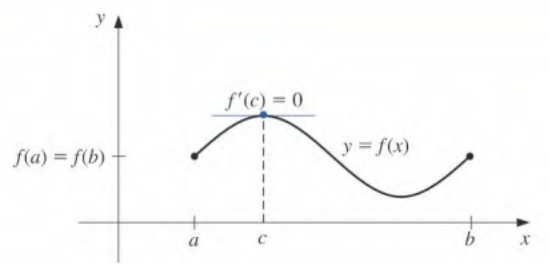
Preliminaries

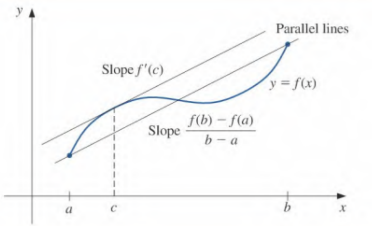
Rolle's Theorem



Source: Thomas Calculus Book

Theorem 1 (Rolle's Theorem)
Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then a number $c \in (a,b)$ exists with $f'(c) = 0$.

Mean Value Theorem

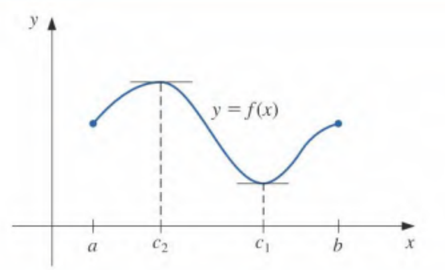


Source: Thomas Calculus Book

Theorem 2 (Mean Value Theorem)
If $f \in C[a,b]$ and f is differentiable on (a,b), then a number $c \in (a,b)$ exists with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Extreme Value Theorem



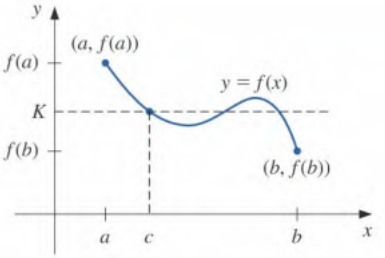
Source: Thomas Calculus Book

Theorem 3 (Extreme Value Theorem)
If $f \in C[a,b]$, then $c_1, c_2 \in [a,b]$ exists with $f(c_1) \le f(x) \le f(c_2)$, for all $x \in [a,b]$. In addition, if f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the end points of $[a,b]$ or where f' is zero.

Generalized Rolle's Theorem

Theorem 4 (Generalized Rolle's Theorem)
Suppose $f \in C[a,b]$ and f is n times differentiable on (a,b). If $f(x) = 0$, at the $n+1$ distinct numbers $a \leq x_0 < x_1 < \cdots < x_n \leq b$, then a number $c \in (x_0, x_n)$ exists with $f^{(n)}(c) = 0$.

Intermediate Value Theorem



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Theorem 5 (Intermediate Value Theorem)
Suppose $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then a number $c \in (a, b)$ exists with $f(c) = K$.

Taylor's Theorem

Theorem 6 (Taylor's Theorem)
Suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ and $R_n(x)$ are called the nth Taylor polynomials and the remainder term or truncation error respectively and are given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

Remarks

Remarks

1. $R_n(x)$ depends on $P_n(x)$.
2. One of the common problem in numerical methods is try to determine a realistic bound for the value of $f^{(n+1)}(\xi(x))$.
3. The infinite series obtained by taking the limit $P_n(x)$ as $n \to \infty$ is called the Taylor series of f about x_0.
4. If $x_0 = 0$, then the Taylor Polynomial is often called Maclaurin polynomial and the Taylor series is often called a Maclaurin series.

Example

Example 7

$$\ln(1+x) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{x^k}{k} + \frac{(-1)^n}{n+1} (1+\xi)^{-(n+1)} x^{n+1}$$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!} e^{\xi}$$

where $x \in [0,1], \xi \in (0,x)$

Riemann Integral

Definition 8 (Riemann Integral)
The Riemann integral of a function f on the interval $[a, b]$ is the following limit, provided it exists:

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to x_0} \sum_{i=1}^{n} f(z_i)\Delta x_i$$

where the numbers $x_0, x_1, \cdots, x_n$ satisfy $a = x_0 \leq x_1 \leq x_1 \leq \cdots \leq x_n = b$, where $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, 2, \cdots, n$ and z_i is arbitrarily chosen in the interval $[x_{i-1}, x_i]$

Riemann Integral of continuous function

Definition 9 (Riemann Integral of continuous function)
A function f that is continuous on an interval $[a, b]$ is also Riemann integrable on $[a, b]$. Therefore, we can select equidistance points in $[a, b]$ and for each $i = 1, 2, \cdots, n$ choose $z_i = x_i$ and we can write the integral as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

where the numbers $x_0, x_1, \cdots, x_n$ satisfy $a = x_0 \leq x_1 \leq x_1 \leq \cdots \leq x_n = b$, and $x_i = a + i\frac{b-a}{n}$

Weighted Mean Value Theorem for Integrals

Theorem 10 (Weighted Mean Value Theorem for Integrals)
Suppose $f \in C[a,b]$, the Riemann integral of g exists on $[a,b]$ and $g(x)$ does not change sign on $[a,b]$. Then there exists a number $c \in (a,b)$ with

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

When $g(x) \equiv 1$, we obtain

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Asymptotic Notations

Big O

Definition 11 (Big O)

Let $\{x_n\}$ and $\{\alpha_n\}$ be two different sequences. We write

$$x_n = O(\alpha_n)$$

if there exist constants C such that

$$|x_n| \leq C|\alpha_n| \quad \forall\, n \geq n_0.$$

If $\alpha_n \neq 0$ for all n, that is,

$$\lim_{n\to\infty} \left|\frac{x_n}{\alpha_n}\right| \leq C$$

Here, we say x_n is big O of α_n.

Little o

Definition 12 (Little o)
We write

$$x_n = o(\alpha_n)$$

if $C = 0$. That is,

$$\lim_{n\to\infty} \left| \frac{x_n}{\alpha_n} \right| = 0$$

Here, we say x_n is little o of α_n.

Let $x_n \to 0$ and $\alpha_n \to 0$.

1. If $x_n = O(\alpha_n)$, then x_n converges to 0 at least as rapidly as α_n.
2. If $x_n = o(\alpha_n)$, then x_n converges to 0 more rapidly than α_n.

Little o and Big O

Example 13

Verify whether the following is true or not?

$$
\begin{aligned}
\frac{n+1}{n^2} &= O\left(\frac{1}{n}\right) \\
\frac{1}{n\ln n} &= o\left(\frac{1}{n}\right) \\
\frac{1}{n} &= o\left(\frac{1}{\ln n}\right) \\
\frac{5}{n} + e^{-n} &= O\left(\frac{1}{n}\right) \\
10\ln(n) + 5(\ln(n))^3 + 7n + 3n^2 + 6n^3 &= O(n^3)
\end{aligned}
$$

Little o and Big O

Example 14

Verify whether the following is true or not?

$$
\begin{aligned}
e^{-n} &= o\left(\frac{1}{n^2}\right) \\
\ln 2 - \sum_{k=1}^{n-1} (-1)^{k-1} \frac{1}{k} &= O\left(\frac{1}{n}\right) \\
e^x - \sum_{k=0}^{n-1} x^k \frac{1}{k!} &= O\left(\frac{1}{n!}\right) \qquad (|x| \leq 1)
\end{aligned}
$$

Thanks

Doubts and Suggestions

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