MA633L-Numerical Analysis

Lecture 20 : Solution of Nonlinear Equations: Complex Roots and Roots of Polynomials

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Nonlinear Equations: Open Methods

Recap: Bracketing Methods

- For bracketing methods, the root is located within an interval.
- Iteratively applying the bracketing methods, we estimate a closer values to the true value of the root.
- These methods converge because they move closer to the truth.
- However, there is a disadvantage that, we have to find the two guesses one for a_0 and one for b_0 which brackets the roots.
- A wrong guess of either a_0 or b_0 will go vain.

Introduction: Open Methods

- In contrast, the open methods require a single starting value or two starting value that do not necessarily bracket the root.
- Open methods converge much faster that bracketing methods.
- However, the disadvantage of open methods is that, it can diver or move away from the true root.

Newton-Raphson Method for Horner’s Method

Newton-Raphson Method

Example 1

Find the approximations to within 10^{-4} to all the real roots of the following polynomial using Newton-Raphson method

$$P_4(x) = x^4 - 18x^3 + 111x^2 - 278x + 240$$

Solution: As per theorem, $r = \frac{240}{518}$ and $R = 279$. Let us start with $x_0 = 0$ in the Newton-Raphson method

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert g(x_i)\rvert$
0	0.000000		240.000000	-278.000000
1	0.863309	1.000000	71.702426	-124.017979
2	1.441471	0.401091	20.315976	-58.216158
⋮	⋮	⋮	⋮	⋮
7	2.000000	0.000006	0.000000	-18.000000

Newton-Raphson Method

The first root is $x_{r_1} = 2$ and the deflated polynomial is

$$Q_3(x) = x^3 - 16x^2 + 79x - 120$$

Again with $x_0 = 1$ on $Q_3(x)$ in the Newton-Raphson method we obtain

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i) \rvert$	$\lvert g(x_i) \rvert$
0	0.000000		-120.000000	79.000000
1	1.518987	1.000000	-33.412367	37.314373
2	2.414416	0.370868	-8.456998	19.226898
3	2.854269	0.154103	-1.609072	12.103952
4	2.987206	0.044502	-0.129084	10.179602
5	2.999887	0.004227	-0.001130	10.001582
6	3.000000	0.000038	-0.000000	10.000000

Newton-Raphson Method

Therefore, the second root is $x_{r_2} = 3$ and the deflated polynomial is

$$R_2(x) = x^2 - 13x + 40$$

and the roots are $x_{r_3} = 5, x_{r_4} = 8$ Hence

$$P_4(x) = (x-2)(x-3)(x-5)(x-8)$$

Remainder Theorem

Example 2

Find the approximations to within 10^{-4} to all the real roots of the following polynomial using Newton-Raphson method

$$P_4(x) = x^4 - 4x^3 + 7x^2 - 5x - 2$$

Solution: As discussed earlier, $r = \frac{2}{9}$, $R = 8$. We begin with 0 again

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i) \rvert$	$\lvert g(x_i) \rvert$
0	0.000000		-2.000000	-5.000000
1	-0.400000	1.000000	1.401600	-12.776000
2	-0.290294	0.377912	0.146322	-10.173223
3	-0.275911	0.052129	0.002258	-9.860299
4	-0.275682	0.000831	0.000001	-9.855368

Newton-Raphson Method

The first root is $x_{r_1} = -0.275682$ and the deflated polynomial is

$$Q_3(x) = x^3 - 4.27568x^2 + 8.1787x - 7.2547$$

Again with $x_0 = 0$ on $Q_3(x)$ in the Newton-Raphson method, we obtain

Iteration (i)	x_i	ϵ_a	$\lvert f(x_i)\rvert$	$\lvert g(x_i)\rvert$
0	0.000000		-7.254731	8.178730
1	0.887024	1.000000	-2.666236	2.953898
2	1.789640	0.504356	-0.580073	2.483300
3	2.023230	0.115454	0.072397	3.157730
4	2.000303	0.011462	0.000931	3.077045
5	2.000000	0.000151	0.000000	3.076001

Newton-Raphson Method

Therefore, the second root is $x_{r_2} = 2$ and the deflated polynomial is

$$R_2(x) = x^2 - 2.27568x + 3.62736$$

and the roots are $x_{r_3} = 1.1378 + 1.5273i, x_{r_4} = 1.1378 - 1.5273i$

Newton-Raphson Method

Note that, in the first example, all roots are real whereas the second example has two complex roots. However, once the polynomial reaches the second degree we can employ the root finding methods for quadratic equation. What will happen if all roots are complex for a polynomial of a degree 4? How can we employ this algorithm when a polynomial of degree 4 or more has at least 4 complex roots? Will it work? Since, our initial guess is always a real number and Newton-Raphson method works around this number, we can find only the real roots. The next section deals with complex roots.

Complex Roots and Müller’s Method

Müller's Method

One problem with applying the Secant, False Position, or Newton's method to polynomials is the possibility of the polynomial having complex roots even when all the coefficients are real numbers. If the initial approximation is a real number, all subsequent approximations will also be real numbers. One way to overcome this difficulty is to begin with a complex initial approximation and do all the computations using complex arithmetic. An alternative approach has its basis in the following theorem.

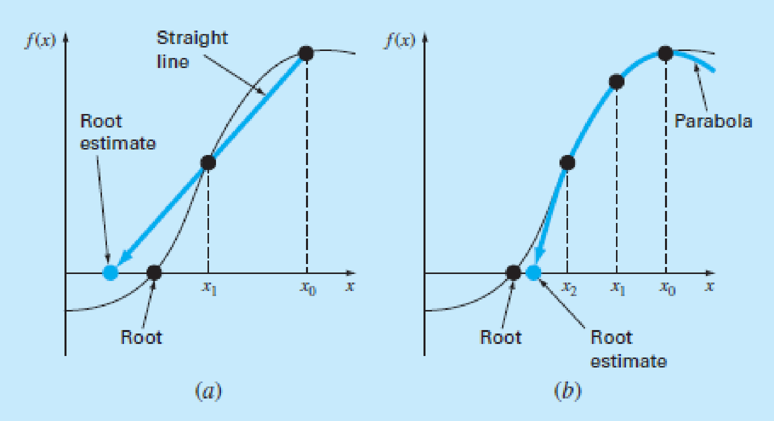
Theorem 3

If $z = a + ib$ is a complex root of multiplicity m of the polynomial $P_n(x)$ with real coefficients, then $z = a - ib$ is also a root of multiplicity m of the polynomial $P_n(x)$, and

$$P_n(x) = (x^2 - 2ax + a^2 + b^2)^m Q(x)$$

Müller’s Method

Secant method obtains a root estimate by projecting a straight line to the x-axis through function values. Müller’s method takes a similar approach but projects a parabola through three points.



Müller's Method

Let us find a parabola equation joining three points: $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$.
Consider the following parabola

$$y = a(x - x_2)^2 + b(x - x_2) + c$$

At given three points

$$f(x_0) = a(x_0 - x_2)^2 + b(x_0 - x_2) + c$$
$$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$
$$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c$$

Müller's Method

On simplification, we obtain

$$c = f(x_2)$$
$$\frac{f(x_0) - f(x_2)}{x_0 - x_2} = a(x_0 - x_2) + b$$
$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = a(x_1 - x_2) + b$$

Let $h_0 = x_1 - x_0, h_1 = x_2 - x_1$

$$\delta_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \text{ and } \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

then

$$a = \frac{\delta_1 - \delta_0}{h_1 + h_0}, \quad b = ah_1 + \delta_1, \quad c = f(x_2)$$

Müller's Method

For,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_2)-f(x_0)}{x_2-x_0}=a(x_1-x_0)$$

$$\frac{(x_2-x_0)(f(x_2)-f(x_1))-(x_2-x_1)(f(x_2)-f(x_0))}{(x_2-x_1)(x_2-x_0)(x_1-x_0)}=a$$

$$\frac{1}{(x_2-x_0)}\frac{f(x_2)(x_2-x_0-x_2+x_1)-f(x_1)(x_2-x_0)+(x_2-x_1)f(x_0)}{(x_2-x_1)(x_1-x_0)}=a$$

$$\frac{1}{(x_2-x_0)}\left[\frac{f(x_2)}{x_2-x_1}+\frac{-f(x_1)(x_2-x_1+x_1-x_0)+(x_2-x_1)f(x_0)}{(x_2-x_1)(x_1-x_0)}\right]=a$$

$$\frac{1}{(x_2-x_0)}\left[\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_1)-f(x_0)}{x_1-x_0}\right]=a$$

Müller's Method

Therefore

$$x - x_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

The formula for Müller's method is given by

$$x_{n+1} = x_n + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}, n \geq 2 \tag{1}$$

This formula gives two possibilities for x_{n+1}, depending on the sign preceding the radical term. In Müller's method, the sign is chosen to agree with the sign of b. Chosen in this manner, root of $P_n(x)$ to x_n. If only real roots are being located, we choose the two original points that are nearest the new root estimate, x_{n+1}. If both real and complex roots are being evaluated, a sequential approach is employed.

Müller's Method

Find the roots of the polynomial. $x^4 - 3x^3 + x^2 + x + 1 = 0$ Let us start with $x_0 = 0.5, x_1 = -0.5, x_2 = 0$ in the Müllers method

Iteration (i)	x_0	$\lvert P_4(x_i)\rvert$
0	0.500000 + 0.000000i	1.437500
1	-0.500000 + 0.000000i	1.187500
2	0.000000 + 0.000000i	1.000000
3	-0.100000 + -0.888819i	3.014896
4	-0.492146 + -0.447031i	0.755895
5	-0.352226 + -0.484132i	0.179524
6	-0.340229 + -0.443036i	0.015964
7	-0.339095 + -0.446656i	0.000112
8	-0.339093 + -0.446630i	0.000000

Müller's Method

The first root is $x_{r_1} = -0.339093 - 0.446630i$ and the second root is $x_{r_2} = -0.339093 + 0.446630i$ If we use $x_0 = 0.5, x_1 = 1.5, x_2 = 1.5$ in the Müller's method

Iteration (i)	x_0	$\lvert P_4(x_i) \rvert$
0	0.500000 + 0.000000i	1.437500
1	1.000000 + 0.000000i	1.000000
2	1.500000 + 0.000000i	0.312500
3	1.406327 + 0.000000i	0.048513
4	1.388783 + 0.000000i	0.001741
5	1.389390 + 0.000000i	0.000003
6	1.389391 + 0.000000i	0.000000

Müller's Method

The third root is $x_{r_3} = 1.38391$ and finally if we use $x_0 = 1.5, x_1 = 2.0, x_2 = 2.5$ in the Müller's method then

Iteration (i)	x_0	$\lvert P_4(x_i) \rvert$
0	1.500000 + 0.000000i	0.312500
1	2.000000 + 0.000000i	1.000000
2	2.500000 + 0.000000i	1.937500
3	2.247332 + 0.000000i	0.245066
4	2.286522 + 0.000000i	0.014464
5	2.288775 + 0.000000i	0.000125
6	2.288795 + 0.000000i	0.000000

The fourth root is $x_{r_3} = 2.88795$

Müller’s Method

From this you can observe that, Müller’s Method can be used to find both complex and real roots. In fact Müller’s method generally converges to the root of a polynomial for any initial approximation choice, although problems can be constructed for which convergence will not occur. For example, suppose that for some i we have $f(x_i) = f(x_{i+1}) = f(x_{i+2}) \neq 0$. The quadratic equation then reduced to a nonzero constant function and never intersects the $x-$axis.

Müller’s Method

The order of convergence of Müller’s method is approximately 1.84.

Accelerating the Convergence

Accelerating the Convergence

- From previous discussions we have observed that bisection method and fixed point methods converge linearly, whereas Newton-Raphson and modified Newton-Raphson method for multiple roots converge quadratically.
- Also, false-position and secant methods converge with the order of convergence as golden ratio.
- Now, using the Aitken's Δ^2 method, we can accelerate the convergence regardless of a sequence that is linearly convergent

Aitken's Δ^2 method

Suppose $\{x_n\}_{n=0}^{\infty}$ is a linearly convergent sequence converging to x_r. We can accelerate the convergence based on the following assumptions:

$x_n - x_r, x_{n+1} - x_r, x_{n+2} - x_r$ are having same signs and for a large n, we have

$$\frac{x_{n+1} - x_r}{x_n - x_r} \approx \frac{x_{n+2} - x_r}{x_{n+1} - x_r}$$

Aitken's Δ^2 method

Then, we have

$$
\begin{aligned}
(x_{n+1} - x_r)^2 &\approx (x_{n+2} - x_r)(x_n - x_r) \\
x_{n+1}^2 - 2x_{n+1}x_r + x_r^2 &\approx x_{n+2}x_n - (x_n + x_{n+2})x_r + x_r^2 \\
(x_{n+2} + x_n - 2x_{n+1})x_r &\approx x_{n+2}x_n - x_{n+1}^2 \\
x_r &\approx \frac{x_{n+2}x_n - x_{n+1}^2}{x_{n+2} + x_n - 2x_{n+1}} \\
&= \frac{x_{n+2}x_n - x_{n+1}^2 - 2x_nx_{n+1} + 2x_nx_{n+1} - x_n^2 + x_n^2}{x_{n+2} + x_n - 2x_{n+1}} \\
&= x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} + x_n - 2x_{n+1}}
\end{aligned}
$$

Aitken's Δ^2 method

Using the Aitken's Δ^2 method, based on the assumption that the sequence $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_n = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} + x_n - 2x_{n+1}} \tag{2}$$

converges more rapidly to x_r than the original sequence $\{x_n\}_{n=0}^{\infty}$ converges to x_r

Aitken's Δ^2 method

Theorem 4

If the sequence $\{x_n\}_{n=0}^{\infty}$ converges to x_r, then the sequence generated by Aitken's method $\{y_n\}_{n=0}^{\infty}$ converges to x_r faster if $x_{n+1} - x_r = (c + \delta_n)(x_n - x_r)$ with $|c| < 1$ and $\lim_{n\to\infty} \delta_n = 0$. That is,

$$\lim_{n\to\infty} \frac{y_n - x_r}{x_n - x_r} = 0$$

Proof: Exercise

One of the important assumption in Aitken's method is that following assumptions: $x_n - x_r, x_{n+1} - x_r, x_{n+2} - x_r$ are having same signs and for a large n. We can relax this assumption and have another assumption

$$\frac{x_{n+1} - x_r}{x_n - x_r} \approx -\frac{x_{n+2} - x_r}{x_{n+1} - x_r}$$

Then extend it further. For more details refer here.

Aitken's Δ^2 method

Now, if we define $\Delta x_n = x_{n+1} - x_n, \quad$ for $n \geq 0$ and $\Delta^2 x_n = \Delta x_{n+1} - \Delta x_n = x_{n+2} - 2x_{n+1} + x_n$, then y_n can be written as

$$y_n = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n} \tag{3}$$

From this we can understand the meaning of Aitken's Δ^2 method.

Steffensen's Method for fixed point iteration

By employing the Aitken's method to a linearly convergent sequence obtained from fixed point iteration, we can accelerate the convergence to quadratic. This procedure is known as Steffensen's method. Suppose we have fixed point iteration

$$x_0, x_1 = f(x_0), x_2 = f(x_1)$$

Then use Aitken's Δ^2 method to compute

$$x_3 = y_0, x_4 = f(x_3), x_5 = f(x_4), x_6 = y_3$$

and repeat the above steps. That is every third term of the Steffensen sequence is generated by the Aitken's Δ^2 method whereas the rest are calculated using fixed-point iteration.

Steffensen's Method for fixed point iteration

Theorem 5

Suppose that $f(x) = x$ has a fixed point x_r with $f'(x_r) \neq 1$. If there exists $\delta > 0$ such that $f \in C^2([x_r - \delta, x_r + \delta])$, then Steffensen's method gives quadratic convergence for any $x_0 \in [x_r - \delta, x_r + \delta]$.

Proof: Exercise.

Accelerating the Newton-Raphson Method

Newton-Raphson method is not applicable when the derivative of any function is not defined. Therefore, the Newton-Raphson method was modified by Steffensen. The Steffensen method for solving the equation $f(x) = 0$ uses the formula

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n)} \tag{4}$$

This method also converges quadratically like Newton-Raphson method however it is free from derivatives.

Accelerating the Newton-Raphson Method

The following modifications on Newton-Raphson method converges cubically

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)} \tag{5}$$

It is called as Halley's Method

Generalized Newton-Raphson Method

A generalized Newton-Raphson method is given by

$$x_{n+1} = x_n - \omega \frac{(f(x_n))^2}{f'(x_n)} \tag{6}$$

where the constant ω is a an acceleration factor chosen to increase the rate of convergence.

Olver's Method

Assume that $f \in C^4[a,b]$, $f(x_r) = 0$, $f'(x_r) \neq 0$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{f''(x_n)f(x_n)^2}{f'(x_n)^3}$$

Other Methods

1. Bairstow's method (Polynomials)
2. Jenkins-Traub Method
3. Laguerre's Method
4. Brent (Hybrid)

Exercises

Exercise 1: Medium

1. Using Newton-Raphson method, compute the root of $\sqrt{2}$ with $x_0 = 1$
2. Find the root of the following equations (with appropriate starting values)
   - 2.1 $2x(1 - x^2 + x)\ln x = x^2 - 1$ in $[0, 1]$
   - 2.2 $\tan x - x = 0$ with $x_0 = 7$
   - 2.3 $e^x - \sqrt{x + 9}$ with $x_0 = 2$
   - 2.4 $x^3 = \sin x + 7$ with $x_0 = 1$
   - 2.5 $\sin x = 1 - x$ with $x_0 = 2$
   - 2.6 $e^{-x} - \cos x = 0$ with $x_0 = \pi/2$

Exercises

Exercise 2: Medium

1. The iteration formula

$$x_{n+1} = x_n - \cos x_n \sin x_n + R \cos^2 x_n$$

where R is a positive constant was obtained by applying Newton method to some function f. What was $f(x)$?

2. Two of the four roots of $x^4 + 2x^3 - 7x^2 + 3$ are positive. Find them by using Newton-Raphson method.
3. What happens if the Newton-Raphson method is applied to $f(x) = \tan^{-1} x$ with $x_0 = 2$? For what values of x_0, Newton-Raphson method converge?

Exercises

Exercise 3: Medium

1. Find the root of the following functions (with appropriate starting values)
   - 1.1 $f(x) = \frac{x}{2} + \frac{1}{x}$ with $x_0 = 1$
   - 1.2 $f(x) = 10 - 2x + \sin x$
   - 1.3 $f(x) = e^x + 1$
   - 1.4 $x^2 - 4\sin x$
   - 1.5 $x^3 + 6x^2 + 11x - 6$

Complex Roots

Exercise 4: Medium

1. Find all the real roots of the following polynomials using Müllers method.
   1.1 $x^3 - 2x^2 + 5$
   1.2 $x^3 + 3x^2 + 1$
   1.3 $x^5 - x^4 + 2x^3 - 3x^2 + x - 4$
   1.4 $x^4 + 5x^3 - 9x^2 - 85x - 136$
   1.5 $x^5 + 11x^4 - 21x^3 - 10x^2 - 21x - 5$
2. Use Bisection, Secant, Newton-Raphson, False-Position together with Müllers methods to find all roots of the following polynomial in the interval $[0.1, 1]$

$$600x^4 - 550x^3 + 200x^2 - 20x - 1$$

Complex Roots

Exercise 5: Medium

1. Aerospace engineers sometimes compute the trajectories of projectiles such as rockets. A related problem deals with the trajectory of a thrown ball. The trajectory of a ball thrown by a right fielder is defined by the (x, y) coordinates as displayed in Figure. The trajectory can be modeled as

$$y = \tan\theta_0 x - \frac{g}{2v_0^2\cos^2\theta_0}x^2 + y_0$$

Find the appropriate initial angle θ_0 , if $v_0 = 30m/$s, and the distance to the catcher is $90m$. Note that the throw leaves the right fielder's hand at an elevation of $1.8m$ and the catcher receives it at $1m$.

Nonlinear Systems

Nonlinear Systems

When you have system of differential equations, we may require to find the equilibrium points. For example, consider Volterra predator-prey equations

$$\dot{x}_1 = x_1 + x_1 x_2$$
$$\dot{x}_2 = x_2 - x_1 x_2$$

Consider the nonlinear circuit equations

$$\dot{x}_1 = v - x_1 + x_2$$
$$\dot{x}_2 = x_1 - f(x_2)$$

System of nonlinear equations

Let us consider the system of n nonlinear equations of the form

$$
\begin{aligned}
f_1(x_1, x_2, \cdots, x_n) &= 0 \\
f_2(x_1, x_2, \cdots, x_n) &= 0 \\
&\cdots \\
f_n(x_1, x_2, \cdots, x_n) &= 0
\end{aligned}
$$

This system of nonlinear equations in n unknowns can be represented by defining a function F mapping $\mathbb{R}^n$ into $\mathbb{R}^n$

$$\mathbf{F}(x_1, x_2, \cdots, x_n) = (f_1(x_1, x_2, \cdots, x_n), f_2(x_1, x_2, \cdots, x_n), \cdots, f_n(x_1, x_2, \cdots, x_n))$$

System of nonlinear equations

Newton-Raphson method can also be applied to system of nonlinear equations. Let $J_{\mathbf{F}}(x)$ denote the Jacobian matrix and it is defined as

$$J_{\mathbf{F}}(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \cdots & \frac{\partial f_3}{\partial x_n} \\ \vdots & \vdots & \vdots \ddots & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{7}$$

Newton-Raphson Method for System of nonlinear equations

If we apply Newton-Raphson method for system of nonlinear equations,

$$\mathbf{F}(\mathbf{x}) = 0$$

the we obtain

$$\mathbf{x}^{n+1} = \mathbf{x}^n - [J_{\mathbf{F}}(\mathbf{x}^n)]^{-1}\mathbf{F}(\mathbf{x}^n)$$

The above equation can also be written as

$$[J_{\mathbf{F}}(\mathbf{x}^n)]\mathbf{x}^{n+1} = [J_{\mathbf{F}}(\mathbf{x}^n)]\mathbf{x}^n - \mathbf{F}(\mathbf{x}^n)$$

Numerical Linear Algebra

The above equation can also be written in the form

$$A\mathbf{x}^{n+1} = b$$

Hence, we apply the Newton-Raphson method for a system of nonlinear equations, you must also learn solving a linear system through numerical linear algebra. In the next lecutre, let us discuss "Numerical Linear Algebra"

Thanks

Doubts and Suggestions

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MA633L-Numerical Analysis

Lecture 20 : Solution of Nonlinear Equations: Complex Roots and Roots of Polynomials

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