MA633L-Numerical Analysis

Lecture 21 : Numerical Linear Algebra - LU Decomposition

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Introduction

Linear System

- One of the important problem in many science and engineering.
- To solve an algebraic linear system

$$Ax = b$$

  for the unknown vector x.
- The coefficient matrix A and the right-hand side vector b are known.
- What is the size of A, x and b?
- Assume: A is $n \times n$, x and b are $n-$dimensional vector.
- The system may or may not have a solution.
- It may have an infinite solutions or unique solution.

Applications of Linear System

- Linear Regression, where A is feature matrix, x is the model parameters and b is the target values
- Ridge Regression (λ: Regularization parameter):

$$\min_x \|Ax - b\|^2 + \lambda\|x\|^2 \implies (A^T A + \lambda I)x = A^T b$$

- PCA: $A^T Av = \lambda v$, $A^T A$ covariance matrix, v principal components, λ variance explained by each component
- SVM: Solve the quadratic optimization for α in dual formulation (Gram matrix)
- Gaussian Process: $(K + \sigma^2 I)y = f$, where K is the kernel matrix, $\sigma^2 I$ noise in observations

Applications of Linear System

- Vandermonde Matrix in Numerical Linear Interpolation
- Hessian Matrix in Newton-Raphson Method, Conjugate Gradient Method
- Numerical PDEs where PDEs are converted to Linear System
- CFD, FEM, Molecular Dynamics, Quantum Chemistry
- Electrical: Control Systems. Electric Circuits, Image processing
- Finance: Portfolio optimization, Risk Management
- Robotics: Inverse Kinematics, Physics-Based Simulations

Linear System

- Gaussian elimination method: one of the standard method for solving the linear system
- Computer or calculator
- School days.
- However, in pure mathematics, the solution is given by

$$x = A^{-1}b$$

  where A^{-1} denotes the inverse of the matrix A.
- But, in most of the real applications, it is not advised to find A^{-1} explicitly, indeed, it is recommended to solve for x.

Linear System

- But, in most of the real applications, it is not advised to find A^{-1} explicitly, indeed, it is recommended to solve for x.
- In applied mathematics, the largest and fastest computers can also face difficulty to solve the system accurately when the number of unknowns are in millions.

Linear System: Important 10 Questions

1. How do we store such a large linear system in the computer?
2. How do we know that the computed answers are correct?
3. How does the computer precision affects the results?
4. Can the algorithm fail?
5. How long will it take to compute the answers?
6. What is the asymptotic operation count of the algorithm?
7. Will the algorithm be stable for perturbation?
8. Can stability be controlled by pivoting?
9. Which strategy of pivoting should be used?
10. How do we know whether the matrix is ill-conditioned?

Diagonal Linear System

Definition 1 (Diagonal linear system)

A matrix A of order n is diagonal if all its nonzero entries are on its diagonal. A diagonal linear system of order n is one whose coefficient matrix is diagonal. That is

$$A_{ij} = \begin{cases} d_{ii} \neq 0 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A diagonal linear system can be represented as

$$\begin{pmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

Diagonal Linear System

Finding the solution of the linear system is easy because each equation determines the value of one unknown, provided that each diagonal entry is nonzero. The solution is given by

$$x_i = \frac{b_i}{d_{ii}}, \quad i = 1, 2, \cdots, n$$

Example 1

Consider the following linear system and find the solution.

$$9x_1 = 3, \;\; 25x_2 = 5, \;\; , 16x_3 = 4, \;\; 36x_4 = 6, \;\; 49x_5 = 7$$

Diagonal Linear System

It is a diagonal linear system and it can be written as

$$\begin{pmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 49 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 6 \\ 7 \end{pmatrix}$$

Without constructing this matrix also we can immediately conclude that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{5} \\ \frac{1}{4} \\ \frac{1}{6} \\ \frac{1}{7} \end{pmatrix}$$

Upper Triangular Linear System

Definition 2 (Upper triangular linear system)
A matrix A of order n is upper triangular if all its nonzero entries are on its diagonal or strictly on the upper triangular entries. An upper triangular linear system of order n is one whose coefficient matrix is upper triangular. That is

$$A_{ij} = \begin{cases} u_{ij} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}$$

An upper triangular linear system can be represented as

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

Backward Substitution

In order to solve an upper triangular linear system, we need a backward substitution which is given by the following algorithm

$$x_i = \frac{y_i - \sum_{j=i+1}^{n} u_{ij} x_j}{u_{ii}}, \quad i = n, n-1, \cdots, 1 \tag{1}$$

Backward Substitution

Example 2

Find the solution of the following linear system.

$$\begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ 34 \\ 25 \\ 10 \end{pmatrix}$$

It is an upper triangular system. Using backward substitution, we obtain that

$$\begin{aligned} x_4 = \frac{10}{10} = 1 \implies x_3 = \frac{25 - 9x_4}{8} &\implies x_3 = \frac{16}{8} = 2 \\ \implies x_2 = \frac{34 - 6x_3 - 7x_4}{5} &\implies x_2 = \frac{15}{5} = 3 \\ \implies x_1 = \frac{20 - 2x_2 - 3x_3 - 4x_4}{2} &\implies x_1 = \frac{4}{2} = 2 \end{aligned}$$

Forward Substitution

Definition 3 (Lower triangular linear system)
A matrix A of order n is upper triangular if all its nonzero entries are on its diagonal or strictly on the lower triangular entries. A lower triangular linear system of order n is one whose coefficient matrix is lower triangular. That is

$$A_{ij} = \begin{cases} l_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

A lower triangular linear system can be represented as

$$\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & u_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

Forward Substitution

In order to solve a lower triangular linear system, one can apply forward substitution. In the backward substitution, we obtain the value of x_n first and then using x_n, we obtain the value of y_{n-1} Using y_n, y_{n-1}, y_{n-2} is calculated. Finally the value of y_1 is calculated. On the other hand, in forward substitution, we obtain the value of x_1 first. Using x_1, we obtain the value of x_2 and so on. The value of

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} y_j}{l_{ii}}, \quad i = 1, 2, \cdots, n \tag{2}$$

LU Decomposition

LU Decomposition

If we can decompose a square matrix A by in the form of

$$A = LU$$

then one can easily obtain the solution of the linear system

$$Ax = b$$

$$Ax = b \implies LUx = b \implies Ly = b \text{ where } Ux = y$$

Since $Ly = b$ is a lower triangular system, by the forward substitution, we can solve for y. Using y in $Ux = y$, which is an upper triangular system, and the backward substitution, we can obtain x.

LU Decomposition

The following steps are used to solve the linear system $Ax = b$.

1. Find L and U such that $A = LU$
2. Using forward substitution, Solve for y from $Ly = b$
3. Use y in $Ux = y$
4. Using backward substitution, Solve for x from $Ux = y$

As per the LU decomposition, we can write A as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

LU Decomposition

Example 3

Obtain the LU decomposition and then obtain the solution x.

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 26 \\ -19 \\ -34 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

As per LU Decomposition

LU Decomposition

Comparing the first entry of LU with A, we obtain that $l_{11}u_{11} = 6$. Now, we can choose any real number l_{11} such that $l_{11}u_{11} = 6$. For the sake of convenience, we always choose $l_{ii} = a_{ii}$, then we obtain that $l_{11} = 6$ and $u_{11} = 1$.

$$l_{11}u_{11} = 6, l_{11} = 6 \implies u_{11} = 1$$

$$l_{11}u_{12} = -2 \implies u_{12} = \frac{-1}{3}$$

$$l_{11}u_{13} = 2 \implies u_{13} = \frac{1}{3}$$

$$l_{11}u_{14} = 4 \implies u_{14} = \frac{2}{3}$$

Since u_{11} is known, we can easily compute the first column of L by the following formula.

$$l_{i1} = \frac{a_{i1}}{u_{11}}, \quad i > 1$$

LU Decomposition

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 12 & l_{22} & 0 & 0 \\ 3 & l_{32} & l_{33} & 0 \\ -6 & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} 1 & -1/3 & 1/3 & 2/3 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Therefore, the formula to compute u_{2j} is given by

$$u_{2j} = \frac{a_{2j} - l_{21}u_{1j}}{l_{22}}, \quad j > 2$$

Since u_{22} and u_{12} are known, we can compute the second column of L by the following formula

$$l_{i2} = \frac{a_{i2} - l_{i1}u_{12}}{u_{22}}, \quad i > 2$$

LU Decomposition

Similarly the formula to compute u_{2j} is given by

$$u_{3j} = \frac{a_{3j} - l_{31}u_{1j} - l_{32}u_{2j}}{l_{33}}, j > 3$$

Since u_{22} and u_{12} are known, we can compute the second column of L by the following formula

$$l_{i3} = \frac{a_{i3} - l_{i1}u_{13} - l_{i2}u_{23}}{u_{33}}, \quad i > 3$$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 12 & -8 & 0 & 0 \\ 3 & -24 & 9 & 0 \\ -6 & 4 & 18 & -18 \end{pmatrix} \begin{pmatrix} 1 & -1/3 & 1/3 & 2/3 \\ 0 & 1/2 & -1/4 & -1/4 \\ 0 & 0 & 2/9 & -5/9 \\ 0 & 0 & 0 & 1/6 \end{pmatrix}$$

LU Decomposition

Now, let us consider the system $Ly = b$

$$\begin{pmatrix} 6 & 0 & 0 & 0 \\ 12 & -8 & 0 & 0 \\ 3 & -24 & 9 & 0 \\ -6 & 4 & 18 & -18 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 26 \\ -19 \\ -34 \end{pmatrix}$$

Using forward substitution, we obtain that

$$y_1 = \frac{16}{6} = \frac{8}{3} \implies 12\left(\frac{8}{3}\right) - 8y_2 = 26 \implies y_2 = \frac{3}{4}$$
$$\implies 3\left(\frac{8}{3}\right) - 24\left(\frac{3}{4}\right) + 9y_3 = -19 \implies y_3 = -1$$
$$\implies -6\left(\frac{8}{3}\right) + 4\left(\frac{3}{4}\right) + 18(1) - 18y_4 = -34 \implies y_4 = \frac{1}{6}$$

LU Decomposition

Now, let us consider the system $Ly = b$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 3/4 \\ -1 \\ 1/6 \end{pmatrix}$$

Now, consider the system $Ux = y$

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 2/3 \\ 0 & 1/2 & -1/4 & -1/4 \\ 0 & 0 & 2/9 & -5/9 \\ 0 & 0 & 0 & 1/6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 3/4 \\ -1 \\ 1/6 \end{pmatrix}$$

LU Decomposition

Using backward substitution, we obtain that

$$x_4 = \frac{1/6}{1/6} = 1 \implies \frac{2}{9}x_3 - \frac{5}{9} = -1 \implies x_3 = -2$$

$$\implies \left(\frac{1}{2}\right)x_2 + \left(\frac{-1}{4}\right)(-2) + \left(\frac{-1}{4}\right)(1) = \left(\frac{3}{4}\right) \implies x_2 = 1$$

$$\implies x_1 - \left(\frac{1}{3}\right) - 2\left(\frac{1}{3}\right) + \frac{2}{3} = \frac{8}{3} \implies x_1 = 3$$

LU Decomposition

For a general LU decomposition, we have first guess $l_{kk} \neq 0$ value or $u_{kk} \neq 0$ values and use the following formula for $j > k$ and $i > k$

$$u_{kj} = \frac{a_{kj} - \sum_{s=1}^{k-1} l_{ks} u_{sj}}{l_{kk}} \tag{3}$$

$$l_{ik} = \frac{a_{ik} - \sum_{s=1}^{k-1} l_{is} u_{sk}}{u_{kk}} \tag{4}$$

From the above two equations, it is easy to note that $l_{kk} \neq 0$ and $u_{kk} \neq 0$ are necessary.

LU Decomposition

Theorem 4

If all n leading principal minors of A are nonsingular, then A has an LU-decomposition.

Hint: Prove by induction. Since $a_{11} \neq 0$, apply the first step in Gaussian elimination and obtain

$$A_2 = L_2 U_2 = \begin{pmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{22} \\ 0 & u_{22} \end{pmatrix}$$

Assume $A_i = L_i U_i, i = 1, 2, \cdots, k-1$. Prove that $A_k = L_k U_k$.

$$\begin{pmatrix} A_{k-1} & c \\ d & a_{kk} \end{pmatrix} = \begin{pmatrix} L_{k-1} & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} U_{k-1} & u \\ 0 & u_{kk} \end{pmatrix}$$

where $c = (a_{1k}, \cdots, a_{k-1k})^t$,$d = (a_{k1}, \cdots, a_{kk-1})^t$,$l = (l_{k1}, \cdots, l_{kk-1})^t$,$u = (u_{1k}, \cdots, u_{k-1k})$ Solve the above system to obtain u_{kk}.

LU Decomposition

How about the converse?

Exercise: Assume that A is invertible. Prove that A has an LU decomposition if and only if all principal minors of A are nonsingular.

Crout, Doolittle, LDL^T, Cholesky

Crout's Decomposition

In LU-decomposition one of the condition specified is that $l_{kk} \neq 0$ and $u_{kk} \neq 0$ are necessary. If we choose, $u_{kk} = 1$, then U is an upper triangular matrix with its diagonal entries as 1. This decomposition is called Crout's decomposition.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & u_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & u_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Doolittle's Decomposition

If we choose, $l_{kk} = 1$, then L is a lower triangular matrix with its diagonal entries as 1. This decomposition is called Doolittle's decomposition.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

Diagonally Dominant

Definition 5 (Diagonally Dominant)

A square matrix $A = (a_{ij})$ is said to be **diagonally dominant** if it satisfies the following inequality:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad 1 \leq i \leq n$$

There is also called row diagonal dominance. Look for column diagonal dominance also.

Gershgorin Disc

Definition 6 (Gershgorin Disc)
The closed disc D_i centered at a_{ii} with radius

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

is called a Gershgorin disc.

$$D_i = \left\{ \lambda : |\lambda - a_{ii}| \leq r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\}, \quad 1 \leq i \leq n$$

Gershgorin Disc Theorem

Theorem 7 (Gershgorin Disc Theorem)
Every eigenvalue of A lies within at least one of Gershgorin disc.

Proof: Let λ be any eigenvalue of A and $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})$ be the corresponding eigenvector. Choose the eigenvector such that

$$\begin{cases} x_{kj} = 1 & j = k \\ |x_{kj}| \leq 1 & j \neq k \end{cases}$$

Such an x always exist as it can be obtained by dividing any eigenvector by its largest modulus. (If not possible, indexing of λ_k can be interchanged).

$$Ax_k = \lambda x_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_{kj} + a_{kk} = \lambda_k$$

Gershgorin Disc Theorem

Hence,

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj}x_{kj} = \lambda - a_{kk}$$

$$\implies |\lambda - a_{kk}| = \left|\sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj}x_{kj}\right| \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}||x_{kj}| \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| = r_k$$

Hence the proof.
Let λ be an eigenvalue of A. Then as per this theorem

$$\lambda \in \bigcup_{i=1}^{n} D_i$$

Gershgorin Disc Theorem

- Now define C_i as follows?

$$C_i = \left\{ \lambda : |\lambda - a_{ii}| \leq r_i = \sum_{\substack{i=1 \\ j \neq i}}^{n} |a_{ij}| \right\}, \quad 1 \leq i \leq n$$

- Is the following statements true?

$$\lambda \in \bigcup_{i=1}^{n} C_i$$

$$\lambda \in \left(\bigcup_{i=1}^{n} D_i \right) \bigcap \left(\bigcup_{i=1}^{n} C_i \right)$$

Gershgorin Disc Theorem

$$\lambda \in \left(\bigcup_{i=1}^{n} D_i\right) \bigcap \left(\bigcup_{i=1}^{n} C_i\right)$$

This statement tightens our bounds on the eigenvalues in some case. Let

$$S_k = \left(\bigcup_{i=1}^{k} D_i\right) \text{ and } S_{n-k} = \left(\bigcup_{i=k+1}^{n-k} D_i\right)$$

such that

$$S_k \bigcap S_{n-k} = \phi$$

then S_k contains exactly k eigenvalues of A.

Gershgorin Disc

If

$$|a_{ii} - a_{jj}| > r_i + r_j, \forall i \neq j$$

then A has distinct eigenvalues each lying in a distinct Gerschgorin disc.
Prove or Disprove.
Is the converse true?
Suppose

$$|a_{ii} - a_{jj}| \leq r_i + r_j$$

for some, i and j, prove or disprove that multiple eigenvalues are in the same disc.

LU Decomposition

Theorem 8

Every diagonally dominant matrix is non-singular and has an LU-decomposition.

Proof: Let A be a diagonally dominant matrix. As per Gershgorin Disc theorem, we have to prove that $\lambda \neq 0$. Suppose $\lambda = 0$, then

$$|a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{ij}|$$

which is a contradiction as we have assumed that A is a diagonally dominant matrix. Also, you can observe that all leading principal minors are nonsingular as you can repeat the above argument using Gershgorin Disc theorem for principal minors. Hence proof follows by Theorem 1.

LU Decomposition

Theorem 9

Let L be unit lower triangular matrix, and U be an upper triangular matrix. Then $A = LU$ is nonsingular if and only if U has no zeros on its diagonal.

Proof: Let $A = LU$ be nonsingular. Since L is unit lower triangular

$$\det(A) = \det(L)\det(U) \implies \det(A) = \det(U) = \prod_{i}^{u_{ii}}$$

Hence the proof.

LDL^T Decomposition

Observe that when Doolittle's decomposition is applied, the first row of U and A are same. Similarly, when Crout's decomposition is applied, then the first column of L and A are same. In fact, we can prove uniqueness of LU decomposition in case of Doolittle and Crout assumptions as follows:

> **Theorem 10**
> Every nonsingular matrix has a unique LU decomposition.

Suppose

$$A = L_1U_1 = L_2U_2 \implies L_1^{-1}L_2 = U_1U_2^{-1}$$

Since left side is lower triangular and right side is upper triangular, it should be a diagonal matrix. Since $diag(L_1^{-1}L_2) = I$ (for Doolittle) or $diag(U_1U_2^{-1}) = I$ (for Crout), it leads to $L_1 = L_2$ and $U_1 = U_2$. Using Gauss-Jordan method it can be obtained easily.

LDL^T Decomposition

If A is a symmetric matrix and A has an LU decomposition, then it has an LDL^T decomposition. For,

$$A = LU \implies A^T = U^T L^T$$
$$A = A^T \implies LU = U^T L^T$$

Since L is unit lower triangular, it is invertible, we can write

$$U = L^{-1} U^T L^T \implies U(L^T)^{-1} = L^{-1} U^T$$

Transpose of an upper triangular matrix is a lower triangular and vice versa. Since L is lower triangular, L^{-1} is also a lower triangular and hence $L^{-1}U^T$ is a lower triangular matrix. On the other side, we have $U(L^T)^{-1}$ is an upper triangular matrix. Therefore,

$$U(L^T)^{-1} = L^{-1} U^T = D \implies U = DL^T \implies A = LDL^T$$

LU Decomposition

When A is symmetric and positive definite and $U = L^T$, then we obtain that

$$A = LL^T$$

Definition 11 (Positive Definite)
A real symmetric matrix A is positive definite if for all nonzero vectors $x \in \mathbb{R}^n$

$$x^T A x > 0$$

- All eigenvalues of a positive definite matrix is strictly positive.
- A is positive definite if and only if all its leading principal minors are positive.
- Positive definite matrix is always nonsingular.

Theorem 12

If A is real, symmetric and positive definite, then it has a unique factorization

$$A = LL^T$$

in which L is lower triangular with positive diagonal.

Since $A > 0, A = A^T$ by LDL^T decomposition, we can obtain

$$A = LDL^T$$

$$A > 0 \implies D > 0 \implies \tilde{L} \equiv LD^{1/2} \implies A = \tilde{L}\tilde{L}^T$$

where the entries of $D^{1/2}$ are $\sqrt{d_{ii}}$.

LU Decomposition for Rectangular Matrices

LU Decomposition

Definition 13 (LU Decomposition)

Given a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. Its LU-decomposition is given by

$$A = LU$$

where $L \in \mathbb{R}^{m \times n}$ is a unit trapezoidal matrix and $U \in \mathbb{R}^{n \times n}$ is a upper triangular with nonzeros on its diagonal.

LU Decomposition

Definition 14 (Principal Leading Submatrix)
Given a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. For $k \leq n$, the $k \times k$ principal leading submatrix of A is a square matrix

$$A_{TL} \in \mathbb{R}^{k \times k}$$

such that

$$A = \begin{bmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{bmatrix}$$

LU Decomposition

Theorem 15 (LU Decomposition)
Given a matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ have linearly independent columns. Then A has a unique LU decomposition if and only if all its principal leading submatrices are nonsingular.

Proof is similar to Theorem 4.

Definition 16 (LU Decomposition)
Given a matrix $A \in \mathbb{R}^{m\times n}$ with $m \leq n$. Its LU-decomposition is given by

$$A = LU$$

where $L \in \mathbb{R}^{m\times n}$ is a unit lower triangular matrix and $U \in \mathbb{R}^{m\times n}$ is a upper triangular.

Thanks

Doubts and Suggestions

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MA633L-Numerical Analysis

Lecture 21 : Numerical Linear Algebra - LU Decomposition

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