MA633L-Numerical Analysis

Lecture 23 : Numerical Linear Algebra -Cholesky Decomposition

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Gershgorin Disc

Gershgorin Disc

Definition 1 (Gershgorin Disc)
The closed disc D_i centered at a_{ii} with radius

$$r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

is called a Gershgorin disc.

$$D_i = \left\{ \lambda : |\lambda - a_{ii}| \leq r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\}, \quad 1 \leq i \leq n$$

Gershgorin Disc Theorem

Theorem 2 (Gershgorin Disc Theorem)
Every eigenvalue of A lies within at least one of Gershgorin disc.

Proof: Let λ be any eigenvalue of A and $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})$ be the corresponding eigenvector. Choose the eigenvector such that

$$\begin{cases} x_{kj} = 1 & j = k \\ |x_{kj}| \leq 1 & j \neq k \end{cases}$$

Such an $\mathbf{x}$ always exist as it can be obtained by dividing any eigenvector by its largest modulus. (If not possible, indexing of λ_k can be interchanged).

$$Ax_k = \lambda x_k = \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj}x_{kj} + a_{kk} = \lambda_k$$

Gershgorin Disc Theorem

Hence,

$$\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_{kj} = \lambda - a_{kk}$$

$$\implies |\lambda - a_{kk}| = \left|\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_{kj}\right| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}||x_{kj}| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}| = r_k$$

Hence the proof.
Let λ be an eigenvalue of A. Then as per this theorem

$$\lambda \in \bigcup_{i=1}^{n} D_i$$

Gershgorin Disc Theorem

- Now define C_i as follows:

$$C_i = \left\{ \lambda : |\lambda - a_{ii}| \leq r_i = \sum_{\substack{i=1 \\ j \neq i}}^{n} |a_{ij}| \right\}, \quad 1 \leq i \leq n$$

- Is the following statements true?

$$\lambda \in \bigcup_{i=1}^{n} C_i$$

$$\lambda \in \left(\bigcup_{i=1}^{n} D_i \right) \bigcap \left(\bigcup_{i=1}^{n} C_i \right)$$

Gershgorin Disc Theorem

$$\lambda \in \left(\bigcup_{i=1}^{n} D_i \right) \bigcap \left(\bigcup_{i=1}^{n} C_i \right)$$

This statement tightens our bounds on the eigenvalues in some case. Let

$$S_k = \left(\bigcup_{i=1}^{k} D_i \right) \text{ and } S_{n-k} = \left(\bigcup_{i=k+1}^{n-k} D_i \right)$$

such that

$$S_k \bigcap S_{n-k} = \phi$$

then S_k contains exactly k eigenvalues of A.

Gershgorin Disc

If

$$|a_{ii} - a_{jj}| > r_i + r_j, \forall i \neq j$$

then A has distinct eigenvalues each lying in a distinct Gerschgorin disc.
Prove or Disprove.
Is the converse true?
Suppose

$$|a_{ii} - a_{jj}| \leq r_i + r_j$$

for some, i and j, prove or disprove that multiple eigenvalues are in the same disc.

LU Decomposition

Theorem 3
Every diagonally dominant matrix is non-singular and has an LU-decomposition.

Proof: Let A be a diagonally dominant matrix. As per Gershgorin Disc theorem, we have to prove that $\lambda \neq 0$. Suppose $\lambda = 0$, then

$$|a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{ij}|$$

which is a contradiction as we have assumed that A is a diagonally dominant matrix. Also, you can observe that all leading principal minors are nonsingular as you can repeat the above argument using Gershgorin Disc theorem for principal minors. Hence proof follows by Theorem 1.

LU Decomposition

Theorem 4
Let L be unit lower triangular matrix, and U be an upper triangular matrix. Then $A = LU$ is nonsingular if and only if U has no zeros on its diagonal.

Proof: Let $A = LU$ be nonsingular. Since L is unit lower triangular

$$\det(A) = \det(L)\det(U) \implies \det(A) = \det(U) = \prod_i u_{ii}$$

Hence the proof.

LDL^T Decomposition

Observe that when Doolittle's decomposition is applied, the first row of U and A are same. Similarly, when Crout's decomposition is applied, then the first column of L and A are same. In fact, we can prove uniqueness of LU decomposition in case of Doolittle and Crout assumptions as follows:

> **Theorem 5**
> Every nonsingular matrix has a unique LU decomposition.

Suppose

$$A = L_1U_1 = L_2U_2 \implies L_1^{-1}L_2 = U_1U_2^{-1}$$

Since left side is lower triangular and right side is upper triangular, it should be a diagonal matrix. Since $diag(L_1^{-1}L_2) = I$ (for Doolittle) or $diag(U_1U_2^{-1}) = I$ (for Crout), it leads to $L_1 = L_2$ and $U_1 = U_2$.

LDL^T Decomposition

If A is a symmetric matrix and A has an LU decomposition, then it has an LDL^T decomposition. For,

$$A = LU \implies A^T = U^T L^T$$
$$A = A^T \implies LU = U^T L^T$$

Since L is unit lower triangular, it is invertible, we can write

$$U = L^{-1} U^T L^T \implies U(L^T)^{-1} = L^{-1} U^T$$

Transpose of an upper triangular matrix is a lower triangular and vice versa. Since L is lower triangular, L^{-1} is also a lower triangular and hence $L^{-1}U^T$ is a lower triangular matrix. On the other side, we have $U(L^T)^{-1}$ is an upper triangular matrix. Therefore,

$$U(L^T)^{-1} = L^{-1} U^T = D \implies U = DL^T \implies A = LDL^T$$

LU Decomposition

When A is symmetric and positive definite and $U = L^T$, then we obtain that

$$A = LL^T$$

Definition 6 (Positive Definite)

A real symmetric matrix A is positive definite if for all nonzero vectors $x \in \mathbb{R}^n$

$$x^T A x > 0$$

- All eigenvalues of a positive definite matrix is strictly positive.
- A is positive definite if and only if all its leading principal minors are positive.
- Positive definite matrix is always nonsingular.

Theorem 7

If A is real, symmetric and positive definite, then it has a unique factorization

$$A = LL^T$$

in which L is lower triangular with positive diagonal.

Since $A > 0$, $A = A^T$ by LDL^T decomposition, we can obtain

$$A = LDL^T$$

$$A > 0 \implies D > 0 \implies \tilde{L} \equiv LD^{1/2} \implies A = \tilde{L}\tilde{L}^T$$

where the entries of $D^{1/2}$ are $\sqrt{d_{ii}}$.

LU Decomposition for Rectangular Matrices (Optional)

LU Decomposition

Definition 8 (LU Decomposition)
Given a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. Its LU-decomposition is given by

$$A = LU$$

where $L \in \mathbb{R}^{m \times n}$ is a unit trapezoidal matrix and $U \in \mathbb{R}^{n \times n}$ is a upper triangular with nonzeros on its diagonal.

LU Decomposition

Definition 9 (Principal Leading Submatrix)

Given a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. For $k \leq n$, the $k \times k$ principal leading submatrix of A is a square matrix

$$A_{TL} \in \mathbb{R}^{k \times k}$$

such that

$$A = \begin{bmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{bmatrix}$$

LU Decomposition

Theorem 10 (LU Decomposition)
Given a matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ have linearly independent columns. Then A has a unique LU decomposition if and only if all its principal leading submatrices are nonsingular.

Proof is similar to Theorem 4.

Definition 11 (LU Decomposition)
Given a matrix $A \in \mathbb{R}^{m\times n}$ with $m \leq n$. Its LU-decomposition is given by

$$A = LU$$

where $L \in \mathbb{R}^{m\times n}$ is a unit lower triangular matrix and $U \in \mathbb{R}^{m\times n}$ is a upper triangular.

Thanks

Doubts and Suggestions

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