MA633L-Numerical Analysis

Lecture 24 : Numerical Linear Algebra - Gaussian Elimination

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Gaussian Elimination

Gaussian Elimination

Do elementary row operation to convert the matrix A to an upper triangular system.

Example 1

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 26 \\ -19 \\ -34 \end{pmatrix}$$

Apply the following row operations $R_2 \to R_2 - 2R_1$ on the second row, $R_3 \to R_3 - \frac{1}{2}R_1$ on the third row and $R_4 \to R_4 + R_1$ on the fourth row. Then we obtain that

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -6 \\ -27 \\ -18 \end{pmatrix}$$

Gaussian Elimination

Now, we try to make the last two entries of the second column to be equal. Applying the following row operations $R_3 \to R_3 - 3R_2$ on the third row and $R_4 \to R_4 + \frac{1}{2}R_2$ on the fourth row produces

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -6 \\ -9 \\ -21 \end{pmatrix}$$

Finally, the row operation $R_4 \to R_4 - 2R_3$ produces

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -6 \\ -9 \\ -3 \end{pmatrix}$$

Gaussian Elimination

The final system is a upper triangular system. Therefore, applying the backward substitution, we get the following result.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

Gaussian Elimination

$$a_{ij} \leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad 1 \leq j \leq n \tag{1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \tag{2}$$

Now, the coefficient of x_1 is the ith equation becomes zero. After this step, we do not need to alter the first equation. Now with second equation as the pivot equation, we repeat the above step for the remaining $n-1 \times n-1$ matrix as follows

$$a_{ij} \leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad 2 \leq j \leq n \tag{3}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \tag{4}$$

Gaussian Elimination

After repeating this, the kth step can be applied on $n-k \times n-k$ matrix with

$$a_{ij} \leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}}\right) a_{kj} \qquad k \leq j \leq n \tag{5}$$

$$b_i \leftarrow b_i - \left(\frac{a_{ik}}{a_{kk}}\right) b_k \tag{6}$$

Here, $\frac{a_{ik}}{a_{kk}}$ is called the multiplier or factor. After applying these steps $n-1$ times, we obtain an upper triangular system. Finally, applying the backward substitution method produces the solution of the system.

Gaussian Elimination Failure

Gaussian Elimination Failure

There are a few cases where Gaussian elimination could fail while implementing in a computer.

Failure 1: Division by Zero

When $a_{kk} = 0$ for some k in the above algorithm, then it can cause division by zero. Problem may also occur when a coefficient is very close to zero.

Failure 2: Roundoff errors

- Increased more digits, then the error might reduce further
- As a rule of thumb, round-off error is more important when the number of equations are 100 or more.

Gaussian Elimination Failure

There are a few cases where Gaussian elimination could fail while implementing in a computer.

Failure 1: Ill-Conditioned

The adequacy of the solution depends on the condition of the system. Well-conditioned systems are those where a small change in one or more of the coefficients results in a similar change in the solution. Ill-conditioned systems are those where small changes in coefficients results in large changes in the solution. Since round-off errors can include small changes in the coefficients of ill-conditioned system, these can produce large solution errors.

Gaussian Elimination Failure

Example 2

$$\begin{pmatrix} 1 & 2 \\ 1.1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 10.4 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

However, consider the following system

$$\begin{pmatrix} 1 & 2 \\ 1.05 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 10.4 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

Singular Systems

Failure 1: Ill-Conditioned

When two are more equations are almost identical, then we obtain ill-conditions. When two or more equations are identical, we obtain a singularity. So, if there are any zero's on the diagonal, we can stop the Gauss elimination by identifying that, the determinant is zero.

Gaussian Elimination as Doolittle

Gaussian Elimination as Doolittle

Obtain the Doolittle decomposition using the Gaussian elimination

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix}$$

For $i = 1$, $u_{11} = 6, u_{12} = -2, u_{13} = 2, u_{14} = 4$ and $l_{11} = 1$. Apply the following row operations $R_2 \to R_2 - 2R_1$ on the second row, $R_3 \to R_3 - \frac{1}{2}R_1$ on the third row and $R_4 \to R_4 + R_1$ on the fourth row. Then we obtain that

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{pmatrix}$$

Gaussian Elimination as Doolittle

Now, $u_{22} = -4, u_{23} = 2, u_{24} = 2$ and $l_{21} = -2, l_{31} = -1/2, l_{41} = 1, l_{22} = 1$ Now, we try to make the last two entries of the second column to be equal. Applying the following row operations $R_3 \to R_3 - 3R_2$ on the third row and $R_4 \to R_4 + \frac{1}{2}R_2$ on the fourth row produces

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{pmatrix}$$

Therefore, $u_{33} = 2, u_{34} = -5$ and $l_{32} = -3, l_{42} = 2, l_{33} = 1$

Gaussian Elimination as Doolittle

Finally, the row operation $R_4 \to R_4 - 2R_3$ produces

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

and $l_{44} = -3$ and $l_{43} = -2, l_{44} = 1$ Therefore the Doolittle decomposition is

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1/2 & 3 & 1 & 0 \\ -1 & 1/2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Operation Count

Gaussian Elimination

$$a_{ij} \leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad 1 \leq j \leq n \tag{7}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \tag{8}$$

Now, the coefficient of x_1 is the ith equation becomes zero. After this step, we do not need to alter the first equation. Now with second equation as the pivot equation, we repeat the above step for the remaining $n-1 \times n-1$ matrix as follows

$$a_{ij} \leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad 2 \leq j \leq n \tag{9}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \tag{10}$$

Operation Count

Step (k)	# Multiplications/Divisions	# Additions/Subtractions
1	$(n-1)(n+1)$	$(n-1)n$
2	$(n-2)(n)$	$(n-2)(n-1)$
3	$(n-3)(n-1)$	$(n-3)(n-2)$
$\vdots$	$\vdots$	$\vdots$
$n-s$	$(n-s)(n-s+2)$	$(n-s)(n-s+1)$
$\vdots$	$\vdots$	$\vdots$
$n-1$	1.3	1.2

Operation Count

Upon summing, we obtain the following number of multiplications/divisions

$$
\begin{aligned}
\sum_{s=1}^{n-1}(n-s)(n-s+2) &= \sum_{s=1}^{n-1}(n-s)^2 + 2\sum_{s=1}^{n-1}(n-s) \\
&= \sum_{s=1}^{n-1} s^2 + 2\sum_{s=1}^{n-1} s \\
&= \frac{(n-1)n(2n-1)}{6} + (n-1)n \\
&= \frac{2n^3+3n^2-5n}{6}
\end{aligned}
$$

Operation Count

Similarly we obtain the number of additions/subtractions as

$$\begin{aligned}
\sum_{s=1}^{n-1}(n-s)(n-s+1) &= \sum_{s=1}^{n-1}(n-s)^2 + \sum_{s=1}^{n-1}(n-s) \\
&= \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} \\
&= \frac{n^3-n}{3}
\end{aligned}$$

Therefore, the total number of operations to convert the linear system to Upper triangular matrix

$$\frac{2n^3+3n^2-5n}{6} + \frac{2n^3-2n}{6} = \frac{4n^3+3n^2-7n}{6}$$

Operation Count

For solving the upper triangular system, we require n^2 operations for backward substitution, therefore, we require $O(n^3)$ operations.

$$\frac{4n^3 + 3n^2 - 7n}{6} + n^2 = \frac{4n^3 + 9n^2 - 7n}{6}$$

n	# Operations
1	1
2	9
3	28
5	115
10	805
50	87025
100	681550
10^6	$6.6667 \times 10^{17} \approx 10^{18}$

Thanks

Doubts and Suggestions

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