MA633L-Numerical Analysis

Lecture 25 : Numerical Linear Algebra - Pivoting

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Elementary Operations

Pivoting

Pivoting

Example 1

$$\begin{pmatrix} 2 & 2c \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2c \\ 2 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

However, the correct solution is $x_1 = x_2 = 1$. On the other hand, if we choose, second row as the pivoting row, then the system becomes

$$\begin{pmatrix} 1 & 1 \\ 2 & 2c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2c \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Pivoting

- The simplest remedy for ill-conditioning is to use more significant figures in the computation.
- If your application can be extended to handle larger word size, such a feature will greatly reduce the problem.
- Obvious problems occur when a pivot element is zero because the normalization step leads to division by zero.
- Problems may also arise when the pivot element is close to, rather than exactly equal to, zero because if the magnitude of the pivot element is small compared to the other elements, and then round-off errors can be introduced.

Pivoting

- Therefore, before each row is normalized, it is advantageous to determine the largest available coefficient in the column below the pivot element.
- The rows can then be switched so that the largest element is the pivot element.
- This is called **partial pivoting.**
- If columns as well as rows are searched for the largest element and then switched, the procedure is called **complete pivoting.**

Pivoting

- Complete pivoting is rarely used because switching columns changes the order of the x's
- Consequently, adds significant and usually unjustified complexity to the computer program.
- Aside from avoiding division by zero, pivoting also minimizes round-off error.
- As such, it also serves as a partial remedy for ill-conditioning.

Pivoting

Example 2

$$\begin{pmatrix} 0.0003 & 3.0000 \\ 1.0000 & 1.0000 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.0001 \\ 1.0000 \end{pmatrix}$$

If we apply Gaussian elimination, we obtain that

$$\begin{pmatrix} 0.0003 & 3.0000 \\ 0 & -99999 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.0001 \\ -6666 \end{pmatrix}$$

$$x_2 = \frac{2}{3}, \quad x_1 = \frac{2.0001 - 3(2/3)}{0.0003}$$

The result is very sensitive to the number of significant figures as shown in the below table

Pivoting

Digits	x_2	x_1
3	0.667	-3.33
4	0.6667	0.0000
5	0.66667	0.30000
6	0.666667	0.330000
7	0.6666667	0.333000

Pivoting

However, consider the following system, where we swapped the first equation and second equation

$$\begin{pmatrix} 1.0000 & 1.0000 \\ 0.0003 & 3.0000 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 2.0001 \end{pmatrix}$$

If we apply Gaussian elimination, we obtain that

$$\begin{pmatrix} 1.0000 & 1.0000 \\ 0.0000 & 2.9997 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.0001 \\ 1.9998 \end{pmatrix}$$

$$x_2 = \frac{2}{3}, \quad x_1 = \frac{1 - (2/3)}{1}$$

Pivoting

Digits	x_2	x_1
0 3	0.667	0.333
4	0.6667	0.3333
5	0.66667	0.33333
6	0.666667	0.333333
7	0.6666667	0.3333333

The pivoting produce much less sensitive results for the round-off errors.

Pivoting

Example 3

$$\begin{pmatrix} 0 & 1.0000 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

When ϵ is introduced in the first entry, we obtain that

$$\begin{pmatrix} \epsilon & 1.0000 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Pivoting

When Gaussian elimination is used we obtain

$$\begin{pmatrix} \epsilon & 1.0000 \\ 0 & 1-\epsilon^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\epsilon^{-1} \end{pmatrix}$$

$$x_2 = \frac{2-\epsilon^{-1}}{1-\epsilon^{-1}} \approx 1$$

$$x_1 = (1-x_2)\epsilon^{-1} \approx 0$$

However, the correct solution is

$$x_2 = \frac{1-2\epsilon}{1-\epsilon} \approx 1$$

$$x_1 = \frac{1}{1-\epsilon} \approx 1$$

Pivoting

When we pivoting we obtain that

$$\begin{pmatrix} 1 & 1 \\ \epsilon & 1.0000 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Upon Gaussian elimination, we get

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 - \epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 - 2\epsilon \end{pmatrix}$$

and

$$x_2 = \frac{1 - 2\epsilon}{1 - \epsilon} \approx 1$$

$$x_1 = \frac{1}{1 - \epsilon} \approx 1$$

Pivoting

- The pivoting strategy is that when $a_{kk} = 0$, the k^{th} must be interchanged with a p^{th} row, where p is the smallest integer greater than k with $a_{pk} \neq 0$.
- To reduce the round-off error it is often necessary to perform row interchanges even when the pivot elements are not zero.
- When a_{kk} after k^{th} operations is small in magnitude compared to a_{jk}, then the magnitude of the factor will be much larger than 1 while computing x_i.
- Round-off error introduced in this factor will contribute to the remaining computation, consequently on each x_k while performing the backward substitution.

Pivoting

Example 4

$$\begin{pmatrix} 0.003 & 59.14 \\ 5.291 & -6.13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 59.17 \\ 46.78 \end{pmatrix}$$

Using Gaussian elimination we obtain

$$\begin{pmatrix} 0.003 & 59.14 \\ 0 & -104300 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 59.17 \\ -104400 \end{pmatrix}$$

and

$$x_2 = \frac{59.14}{0.003} \approx 2000, \quad x_1 \approx -10$$

This ruins the approximation as the true value is $x_1 = 10.000$. However, for larger systems, it is not easy to predict the devastating round-off error in advance.

Pivoting

- The partial pivoting in general uses the following method to overcome this issue.
- Select an element a_{pq} with larger magnitude as the pivot elements after the k^{th} row operations.
- Interchange the k^{th} row and p^{th} row. That is, we determine $p \geq k$ such that

$$a_{pk} = \max_{k \leq i \leq n} |a_{ik}|$$

- Perform $R_k \leftrightarrow R_p$.

Pivoting

Example 5

For the above example,

$$\max_{1 \leq i \leq 2} |a_{i1}| = \max\{0.003, 5.291\} = 5.291$$

Therefore, interchange R_1 and R_2, the system becomes

$$\begin{pmatrix} 5.291 & -6.13 \\ 0.003 & 59.14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 46.78 \\ 59.17 \end{pmatrix}$$

Using Gaussian elimination and backward substitution, we obtain that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 1.0 \end{pmatrix}$$

Scaled Partial Pivoting

Scaled Partial Pivoting

- Simply picking the largest number in magnitude is done in partial pivoting may work well.
- In this case, row scaling does not play a role, that is relative sizes of entries in a row are not considered.
- In certain situations, the simple partial pivoting may not work.
- Consider the following example

Scaled Partial Pivoting

Example 6

$$\begin{pmatrix} 2 & 2c \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2c \\ 2 \end{pmatrix}$$

Now,

$$\max_{1 \le i \le 2} |a_{i1}| = \max\{2, 1\} = 2$$

Therefore, no interchange is required and by Using Gaussian elimination and backward substitution, we obtain that

$$\begin{pmatrix} 2 & 2c \\ 0 & 1-c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2c \\ 2-c \end{pmatrix}$$

Scaled Partial Pivoting

Suppose our c is so large such that $1 - c \approx -c$ and $2 - c \approx -c$, then

$$\begin{pmatrix} 2 & 2c \\ 0 & -c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2c \\ -c \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Scaled Partial Pivoting

However, the correct solution is $x_1 = x_2 = 1$. On the other hand, if we choose, second row as the pivoting row, then the system becomes

$$\begin{pmatrix} 1 & 1 \\ 2 & 2c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2c \end{pmatrix}$$

and then Gaussian elimination produces

$$\begin{pmatrix} 1 & 1 \\ 0 & 2c - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2c - 4 \end{pmatrix}$$

Scaled Partial Pivoting

Suppose our c is so large such that $2c - 2 \approx 2c$ and $2c - 4 \approx 2c$, then

$$\begin{pmatrix} 1 & 1 \\ 0 & 2c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2c \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Scaled Partial Pivoting

Remarks

- This example illustrates that the order in which we treat the equations significantly affects the accuracy of the elimination algorithm in the computer.
- In the naive Gaussian elimination algorithm, we use the first equation to eliminate x_1 from the rest of $n-1$ equations.
- Then we use the second equations to eliminate x_2 from the following $n-2$ equations.
- We follow the natural order in it.
- The last is not used as an operating equation in the naive elimination, at no time are factors of it subtracted from other equations.

Scaled Partial Pivoting

- Therefore, a new strategy is required for selecting the pivot row and pivot element.
- The best approach is complete pivoting as it searches over all entries, but that it is an expensive method.
- However, partial pivoting searches only the column entries.
- We advocate a strategy that simulates a scaling of the row vectors and then selects a pivot element the relatively largest entry in a column.
- Also, rather than interchanging rows to move the desired element in the pivot position, we use an indexing array to avoid the data movement.

Scaled Partial Pivoting

Example 7

$$\begin{pmatrix} 30 & 591400 \\ 5.291 & -6.13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 591700 \\ 46.78 \end{pmatrix}$$

This is same as example 9.14 but first row is multiplied by 10^4. However, it leads to the same inaccurate solution

$$\begin{pmatrix} 30 & 591400 \\ 0 & -104300 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 591700 \\ -104400 \end{pmatrix}$$

and

$$x_2 \approx 1.001$$

$$x_1 \approx -10$$

Scaled Partial Pivoting

Scaled partial pivoting is needed for the above example. It places the element in the pivot position that is the largest relative to the entries in its row. The first step in this procedure is to define a scalar s_i for each row as

$$s_i = \max_{1 \le j \le n} |a_{ij}|$$

Scaled Partial Pivoting

If $s_i = 0$ for some i, then the system has no unique solution as all entries in ith row are 0. Therefore, assume that $s_i \neq 0$. Therefore, we found an appropriate row interchange to place zeros in the first column. This is found by choosing the least integer p such that

$$\frac{a_{p1}}{s_p} = \max_{1 \leq k \leq n} \frac{a_{k1}}{s_k}$$

and then do the row operations $R_1 \leftrightarrow R_p$.

Scaled Partial Pivoting

The effect of scaling is to ensure that the largest element in each row has a relative magnitude of 1 before the comparison for row interchange is performed. In a similar fashion, before eliminating the variable x_i using the operations

$$R_k - \frac{a_{ki}}{a_{ii}} R_i, k = i + 1, \cdots, n$$

we select the smallest integer $p \geq i$ with

$$\frac{|a_{pi}|}{s_p} = \max_{1 \leq k \leq n} \frac{|a_{ki}|}{s_k}$$

and perform $R_i \leftrightarrow E_p$ if $i \neq p$. The scale factors $s_1, s_2, \cdots, s_n$ are computed only once at the start of the procedure. They are row dependent, so they must also be interchanged when row interchanges are performed.

Scaled Partial Pivoting

Example 8

$$\begin{pmatrix} 30 & 591400 \\ 5.291 & -6.13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 591700 \\ 46.78 \end{pmatrix}$$

In this example,

$$s_1 = \max\{|30|, |591400|\} = 591400$$

$$s_2 = \max\{|5.291|, |-6.13|\} = 6.13$$

Consequently,

$$\frac{|a_{11}|}{s_1} = \frac{30}{591400} = 0.5073 \times 10^{-4}$$

$$\frac{|a_{21}|}{s_2} = \frac{5.291}{6.13} = 0.8631$$

Scaled Partial Pivoting

Interchange $R_1 \leftrightarrow R_2$, then

$$\begin{pmatrix} 5.291 & 6.13 \\ 30 & 591400 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 46.78 \\ 591700 \end{pmatrix}$$

and

$$x_2 \approx 1.00$$

$$x_1 \approx 10$$

This produces the correct result.

Scaled Partial Pivoting

Example 9

$$\begin{pmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -34 \\ 16 \\ 26 \end{pmatrix}$$

Scaled Partial Pivoting

In this example,

$$s_1 = 13, s_2 = 18, s_3 = 6, s_4 = 12$$

Let $s = [13, 18, 6, 12]$. Consequently,

$$\frac{|a_{11}|}{s_1} = \frac{3}{13} \approx 0.23, \quad \frac{|a_{21}|}{s_2} = \frac{6}{18} \approx 0.33, \quad \frac{|a_{31}|}{s_3} = \frac{6}{6} = 1.0, \quad \frac{|a_{41}|}{s_4} = \frac{12}{12} = 1.0$$

The largest value occur at R_3, therefore interchange $R_3 \leftrightarrow R_1$ and then $s = [6, 18, 13, 12]$

Scaled Partial Pivoting

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 12 & -8 & 6 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -34 \\ -19 \\ 26 \end{pmatrix}$$

Now apply $R_2 \to R_2 - \frac{1}{2}R_1, R_3 \to R_3 + R_1, R_4 \to R_4 - 2R_1$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 0 & -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -27 \\ -18 \\ -6 \end{pmatrix}$$

Scaled Partial Pivoting

Now,

$$\frac{|a_{22}|}{s_2} = \frac{12}{18} \approx 0.666, \quad \frac{|a_{32}|}{s_3} = \frac{2}{13} \approx 0.15, \quad \frac{|a_{42}|}{s_4} = \frac{4}{12} \approx 0.33$$

Therefore, no row interchange required now, we do the row operations as follows. $R_3 \to R_3 + \frac{1}{6}R_2, R_4 \to R_4 - \frac{1}{3}R_2$,

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 0 & 0 & -2/3 & 5/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -27 \\ -45/2 \\ 3 \end{pmatrix}$$

Scaled Partial Pivoting

Now,

$$\frac{|a_{33}|}{s_3} = \frac{13/3}{13} \approx 0.333, \quad \frac{|a_{43}|}{s_4} = \frac{2/3}{12} \approx 0.055$$

Again, no interchange is required, and we apply $R_4 \to R_4 + \frac{2}{13}R_3$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 0 & 0 & 0 & -6/13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ -27 \\ -45/2 \\ -6/13 \end{pmatrix}$$

Scaled Partial Pivoting

Using backward substitution, we obtain that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

Theorem

Theorem 10

Gaussian elimination without pivoting preserves the diagonal dominance of a matrix.

Gauss-Jordan

Gauss-Jordan

The Gauss-Jordan method is a variation of Gauss elimination method. The major difference is that when an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations, rather than just the subsequent ones. In addition, all rows are normalized by dividing them by their pivot elements. Thus, the elimination results in an identity matrix rather than a triangular matrix. Therefore, back substitution method is not necessary.

Example 11

$$\begin{pmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19 \\ -34 \\ 16 \\ 26 \end{pmatrix}$$

Gauss-Jordan

Row operations $R_2 \to R_2 - (-6)R_1, R_3 \to R_3 - 6R_1, R_4 \to R_4 - 12R_1$, we obtain that

$$\begin{pmatrix} 1 & -13/3 & 3 & 1 \\ 0 & -22 & 19 & -12 \\ 0 & 24 & -16 & -2 \\ 0 & 44 & -30 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -19/3 \\ -72 \\ 54 \\ 102 \end{pmatrix}$$

Normalize the second row, by dividing by the pivot element, that is, $R_2 \to \frac{1}{-22}R_2$. Later using the row operations $R_1 \to R_1 - \frac{-13}{3}R_2$, $R_3 \to R_3 - 24R_2, R_4 \to R_4 - 44R_2$, we obtain that

$$\begin{pmatrix} 1 & 0 & -49/66 & 222/66 \\ 0 & 1 & -19/22 & 12/22 \\ 0 & 0 & 104/22 & -332/22 \\ 0 & 0 & 8 & -26 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 518/66 \\ 72/22 \\ -540/22 \\ -42 \end{pmatrix}$$

Gauss-Jordan

Normalize the third row, by dividing by the pivot element, that is, $R_3 \to \frac{22}{104} R_2$.
Later using the row operations
$R_1 \to R_1 - \frac{-49}{66} R_3, R_2 \to R_2 - \frac{-19}{22} R_3, R_4 \to R_4 - 8R_3$, we obtain that

$$\begin{pmatrix} 1 & 0 & 0 & 6820/6864 \\ 0 & 1 & 0 & -5060/2288 \\ 0 & 0 & 1 & -332/104 \\ 0 & 0 & 0 & -48/104 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 27412/6864 \\ 2772/2288 \\ -540/104 \\ -48/104 \end{pmatrix}$$

Gauss-Jordan

Normalize the fourth row, by dividing by the pivot element, that is, $R_4 \to \frac{-48}{104} R_2$. Later using the row operations $R_1 \to R_1 - \frac{6820}{6864} R_4$, $R_2 \to R_2 - \frac{-5060}{2288} R_4$, $R_3 \to R_3 - \frac{-332}{104} R_4$, we obtain that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in

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Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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