MA633L-Numerical Analysis

Lecture 27 : Numerical Linear Algebra - Matrix Norms

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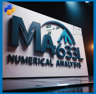

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Recap

Iterative Schemes

The underlying principles behind iterative methods to solve $Ax = b$ are as follows:

1. Guess any $x^{(0)}$
2. Compute the residual $r^{(0)} = b - Ax^{(0)}$
3. Compute the residual norm, that is $\|r^{(0)}\|$
4. Use an algorithm to compute $x^{(1)}$ involving A, b and $x^{(0)}$
5. Recompute the residual norm $\|r^{(1)}\| = \|b - Ax^{(1)}\|$
6. For given $x^{(i)}$, iterate this process until you get the residual as zero or an equivalent condition

Norms

Definition 1 (Vector Norms)
A vector norm, usually denoted by $\|.\|$, is a function from a vector space V to the set of nonnegative real numbers that obeys the following three postulates. $\|.\|: V \to \mathbb{R}_+$ such that

$$\begin{aligned} \|\mathbf{x}\| > 0 \quad & \text{if } \mathbf{x} \neq 0, \mathbf{x} \in V \\ \|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\| \quad & \text{if } \alpha \in \mathbb{R}, \mathbf{x} \in V \\ \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad & \text{if } \mathbf{x}, \mathbf{y} \in V \end{aligned}$$

Matrix Norms

Matrix Norms

Definition 2 (Matrix Norm)
A matrix norm on the set of all $n \times n$ matrices, usually denoted by $\|.\|$, is a function from a $M_{n\times n}(\mathbb{R})$ to the set of nonnegative real numbers that obeys the following three postulates. $\|.\| : M_{n\times n}(\mathbb{R}) \to \mathbb{R}_+$ such that

$$\|A\| \geq 0 \quad \text{(nongativity)}$$

$$\|A\| = 0, \text{ if and only if A is a zero matrix} \quad \text{(Mapping of the Identity)}$$

$$\|\alpha A\| = |\alpha|\|A\| \quad \text{if } \alpha \in \mathbb{R} \quad \text{(Scalar multiplication)}$$

$$\|A + B\| \leq \|A\| + \|B\| \quad \text{(Triangle Inequality)}$$

$$\|AB\| \leq \|A\|\|B\| \quad \text{(Consistency)}$$

where $A, B \in M_{n\times n}(\mathbb{R})$

Matrix Norm

If $A = (a_{ij})$ is an $n \times n$ matrix and $\|.\|$ is a vector norm in $\mathbb{R}^n$, then the matrix norm is

$$\|A\| = \max_{\|x\|=1} \|Ax\| \qquad \text{(Vector Induced Norm)}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \qquad \text{(Row Sum Norm)}$$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| \qquad \text{(Column Sum Norm)}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2} \qquad \text{(Frobenius Norm)}$$

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sigma_{max}(A)$$

Matrix Norm

- The $\|A\|_1$ norm is also called as the **column sum norm** as it computes the maximum absolute column sum of the matrix.
- The $\|A\|_\infty$ norm is also called as the **row sum norm** as it computes the maximum absolute row sum of the matrix.
- An induced matrix norm is also called as **operator norm**
- The $\|A\|_2$ norm is also called as the **spectral norm**
- For Symmetric matrix $\|A\|_1 = \|A\|_\infty$ and $\|A\|_2 = \rho(A)$
- $\|x\|_2$ norm is usually called as Euclidean norm, but $\|A\|_2$ is not.
- If Q is orthogonal, then $\|Q\|_2 = 1$ and hence an orthogonal matrix is also called an isometric matrix.

Theorem

The distance between two matrices $A, B \in M_{n\times n}(\mathbb{R})$ is $\|A - B\|$.

Theorem 3

If $\|.\|$ is a vector norm in $\mathbb{R}^n$, then the vector induced norm is a matrix norm

Proof: It is enough to prove that $\|.\|$ is a matrix norm. If $A \neq 0$, at least one of its column is not a zero vector. Let jth column $(A)_j \neq 0$. Then

$$\|A\| = \max_{\|x\|=1} \|Ax\| \geq \|Ae_k\| > 0$$

$$\|\alpha A\| = \max_{\|x\|=1} \|\alpha Ax\| = \max_{\|x\|=1} |\alpha|\|Ax\| = |\alpha| \max_{\|x\|=1} \|Ax\| = |\alpha|\|A\|$$

Theorems and Proof

Now,

$$\|A+B\| = \max_{\|x\|=1} \|(A+B)x\| = \max_{\|x\|=1} \|Ax+Bx\| \leq \max_{\|x\|=1} (\|Ax\| + \|Bx\|)$$

$$\leq \max_{\|x\|=1} \|Ax\| + \max_{\|x\|=1} \|Bx\| = \|A\| + \|B\|$$

$$\|Ax\| \leq \|A\|\|x\|$$

Let x be any arbitrary vector, choose $y = \frac{x}{\|x\|}$, then

$$\|Ax\| = \|A(y\|x\|)\| = \|x\|\|Ay\| \implies \|Ax\| \leq \|x\| \max_{\|y\|=1} \|Ay\| = \|A\|\|x\|$$

$$\|AB\| = \max_{\|x\|=1} \|ABx\| \leq \max_{\|x\|=1} \|A\|\|Bx\| \leq \max_{\|x\|=1} \|A\|\|B\|\|x\| = \|A\|\|B\|$$

Hence the proof.

Theorems and Proof

Theorem 4

If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

Proof: Observe that when $Ax = b$

$$b_i = \sum_{j=1}^{n} a_{ij} x_j \implies \|b\|_\infty = \max_{1 \le i \le n} |b_i| = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

Theorems and Proof

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\| = \max_{\|x\|_\infty=1} \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}x_j| \le \max_{\|x\|_\infty=1} \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}||x_j|$$

$$\implies \|A\|_\infty \le \left(\max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \right) \max_{\|x\|_\infty=1} |x_j| = \left(\max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \right) \max_{\|x\|_\infty=1} \|x\|$$

$$\implies \|A\|_\infty \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

It is enough to prove that the equality is achieved for some x. Choose your x such that $x_j = sign(A_{ij})$.

Theorems and Proof

Then the proof follows for row sum

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

For column sum, consider the columns of A as $A_1, A_2, \cdots, A_n$

$$\|Ax\|_1 = \left\|\sum_{i=1}^{n} A_i x\right\|_1 \leq \sum_{i=1}^{n} \|A_i x\|_1 = \sum_{i=1}^{n} |x| \|A_i\|_1 \leq \max_{1\leq j\leq n} \|x\|_1 \sum_{i=1}^{n} |a_{ij}|$$

$$\implies \|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 \leq \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$$

The equality and the proof follows by taking the standard basis vector for x.

Frobenius Norm

The Frobenius norm is given by $\|A\|_F$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2}$$

Theorems and Proof

Theorem 5

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = \textbf{trace}(A^T A)$$

$$\|A\|_F^2 = \textbf{trace}(A^T A)$$

Proof: If $A = (a_{ij})$ and $B = (b_{ij})$, then $C = AB = (c_{ij})$ is given by, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \implies c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

Theorems and Proof

When $C = A^T A$, we have

$$c_{ii} = \sum_{k=1}^{n} a_{ki} a_{ki} = \sum_{k=1}^{n} a_{ki}^2 = \sum_{k=1}^{n} a_{ik}^2 = \sum_{j=1}^{n} a_{ij}^2$$

$$\textbf{trace}(A^T A) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2$$

the proof follows.

Theorems and Proof

Theorem 6

Frobenius norm is a norm.

Proof: If $A \neq 0$, at least one element is not a zero element. Let it be a_{pk}. Hence

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2} \geq |a_{pk}| > 0$$

$$\|\alpha A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha a_{ij}|^2} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha||a_{ij}|^2} = |\alpha|\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2} = |\alpha|\|A\|_F$$

Theorems and Proof

$$\|A+B\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}+b_{ij}|^2$$

We know

$$|a+b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$$

Hence

$$\|A+B\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||b_{ij}|$$

$$\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||b_{ij}|$$

Theorems and Proof

By Cauchy-Schwarz inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||b_{ij}| \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2} = \|A\|_F\|B\|_F$$

$$\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + 2\|A\|_F\|B\|_F \implies \|A+B\|_F \leq \|A\|_F + \|B\|_F$$

$$\|AB\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{k=1}^{n} a_{ik}b_{kj}\right|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|a_{ik}|^2\sum_{k=1}^{n}|b_{kj}|^2\right)$$

$$\|AB\|_F^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|a_{ik}|^2\sum_{l=1}^{n}|b_{lj}|^2\right) = \sum_{i=1}^{n}\sum_{k=1}^{n}|a_{ik}|^2\sum_{l=1}^{n}\sum_{j=1}^{n}|b_{lj}|^2 = \|A\|_F^2\|B\|_F^2$$

Theorems and Proof

Theorem 7

If U and V are orthogonal, then

$$\|UA\|_F = \|AV\|_F = \|A\|_F$$

Proof:

$$\|UA\|_F^2 = \textbf{trace}((UA)^T(UA)) = \textbf{trace}(A^TU^TUA) = \textbf{trace}(A^TA) = \|A\|_F^2$$

Since $\textbf{trace}(A^TA) = \textbf{trace}(AA^T)$, we get

$$\|AV\|_F^2 = \textbf{trace}((AV)^T(AV)) = \textbf{trace}((AV)(AV)^T)$$

$$= \textbf{trace}(AVV^TA^T) = \textbf{trace}(AA^T) = \textbf{trace}(A^TA) = \|A\|_F^2$$

Example

Example 8

Compute the $\|.\|_1$, $\|.\|_F$ and $\|.\|_\infty$ norms of the following matrix.

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix}$$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| = \max\{27, 27, 18, 35\} = 35$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = \max\{14, 36, 28, 29\} = 36$$

$$\|A\|_F = 32.3883$$

Eigenvalues

If I denotes the identity matrix, then $\|I\| = 1$. (True or False)

Definition 9 (Characteristic Polynomial:)
If A is a square matrix, the characteristic polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I)$$

Definition 10 (Eigenvalue and Eigenvector)
If p is a characteristic polynomial of A, the roots of p are called eigenvalues or characteristic values of A. If λ is an eigenvalue of A and $x \neq 0$ such that $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector of A.

Spectral Radius

Usually, the matrix is considered over the field C though the matrix entries can be real.

Definition 11 (Singular Values)

The singular values σ of an $m \times n$ matrix A are the positive square roots of the nonzero eigenvalues of the $n \times n$ symmetric matrix $A^T A$.

$$\sigma_i(A) = \sqrt{\lambda_i(A^T A)}$$

Definition 12 (Spectral Radius)

The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$$

This is also denoted as $\lambda_{max}(A)$.

Spectral Radius

The matrix norm induced by the Euclidean norm (ℓ_2 norm) is the spectral norm, and is given by

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sigma_{max}(A)$$

This is the minimum norm and provides the tightest measure of size.

Theorem 13

If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2,$$

$$\|A\|_F = \sqrt{\text{trace}(A^T A)}$$

$$\rho(A) \leq \|A\|$$

for any natural norm $\|.\|$.

Theorem

Definition 1 (Convergent)

We call a matrix A as convergent if

$$\lim_{k \to \infty} (A^k)_{ij} = 0$$

for all i, j

Example 14

$$A = \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix}, A^2 = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{pmatrix}, A^3 = \begin{pmatrix} 1/8 & 0 \\ 3/16 & 1/8 \end{pmatrix},$$

$$A^n = \begin{pmatrix} 1/2^n & 0 \\ n/2^{n+1} & 1/2^n \end{pmatrix}$$

Therefore, the matrix A is convergent.

Theorem

Theorem 15

The following statements are equivalent:

1. A is a convergent matrix.
2. $\lim_{n\to\infty} \|A^n\| = 0$ for some natural norm.
3. $\lim_{n\to\infty} \|A^n\| = 0$ for all natural norm.
4. $\rho(A) < 1$.
5. $\lim_{n\to\infty} A^n x = 0$ for every x.

Theorems

Theorem 16

Suppose

$$Ax = b$$

where A is an $n \times n$ matrix and A is invertible. If A^{-1} is perturbed to obtain a new matrix B and the solution $x = A^{-1}b$ is perturbed to become a new vector $\tilde{x} = Bb$, then

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \|I - BA\|$$

Proof:

$$\|x - \tilde{x}\| = \|x - Bb\| = \|x - BAx\| = \|(I - BA)x\| \leq \|I - BA\|\|x\|$$

Hence the Proof.

Theorems

The above theorem gives an upper bound on $\frac{\|x-\tilde{x}\|}{\|x\|}$ and this ratio denotes the relative error.

Example 17

Suppose that the vector b is perturbed to obtain a vector $\tilde{b}$. If x and $\tilde{x}$ satisfy $Ax = b$ and $A\tilde{x} = \tilde{b}$, by how much do x and $\tilde{x}$ differ in absolute and relative terms?

Solution:

$$\|x - \tilde{x}\| = \|A^{-1}b - A^{-1}\tilde{b}\| = \|A^{-1}(b - \tilde{b})\| \leq \|A^{-1}\|\|b - \tilde{b}\|$$

$$b = Ax \implies \|b\| = \|A\|\|x\| \implies \frac{1}{\|x\|} = \frac{\|A\|}{\|b\|}$$

Hence,

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|b - \tilde{b}\|}{\|b\|}$$

Condition Number

Condition Number

The above inequality implies that the relative error in x is not greater than $\|A\|\|A^{-1}\|$ times the relative error in b.

Definition 18 (Condition Numbmer)
If A is an invertible matrix, then its condition number $\kappa(A)$ is defined by

$$\kappa(A) = \|A\|\|A^{-1}\| \tag{1}$$

From the equation (1), if the condition number is small, then perturbations in b lead to small perturbation in $\mathbf{x}$. The inequality $k(A) \geq 1$ is always true (Why?!).

Condition Number

1. From equation (1) from equation it seems that we need to compute the inverse of A to obtain the condition number if the norm is used as ℓ_2-matrix norm.
2. Also, it can be shown that the condition number $\kappa(A)$ gauges the transfer of error from the matrix A and the vector b to the solution $\mathbf{x}$.
3. If $\kappa(A) = 10^k$, then one can expect to lose at least k digits of precision in solving the system $A\mathbf{x} = \mathbf{b}$.

Condition Number

1. If the matrix A is very sensitive to perturbations in the elements of A, or to perturbations of the components of b, then this fact is reflected in A having a large condition number. Such matrix is called **ill-conditioned**. That is, larger the condition number, more ill-conditioned system.
2. True or False: When a matrix A is invertible, then $\kappa(A) = \kappa(A^{-1})$. For identity matrix $\kappa(I) = 1$. If α is a scalar, then $\kappa(\alpha A) = \kappa(A)$
3. When the condition number is near 1, it is an well-conditioned matrix whereas large values indicate that, it is an ill-condition matrix.
4. When the condition number of a matrix is very large, one can suspect that the model and numerical results.

Condition Number

Example 19

Hilbert Matrix: A Hilbert matrix H_n is defined as follows:

$$a_{ij} = \frac{1}{i+j-1} \tag{2}$$

$$H_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

The condition number of the matrix H_4 is $\kappa(H_4) = 1551.4$ and its determinant is 1.65×10^{-7}. It is an ill-conditioned matrix.

Condition Number

If A is singular, it is customary to define $\kappa(A) = \infty$.

Example 20

Find the condition number of the following matrix with respect to $\ell_1, \ell_2, \ell_\infty$ and Frobenius norm

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 3 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

Condition Number

$$A^T A = \begin{pmatrix} 14 & -5 & 15 \\ -5 & 2 & -5 \\ 15 & -5 & 18 \end{pmatrix}$$

$$A^{-T} A^{-1} = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 35 & -9 \\ -1 & -9 & 3 \end{pmatrix}$$

$$\lambda(A^T A) = \{0.1070, 1.1411, 32.7519\}$$

$$\lambda(A^{-T} A^{-1}) = \{0.0305, 0.8764, 9.3431\}$$

$$\|A\|_1 = \max\{6, 2, 6\} = 6$$

$$\|A\|_2 = \sqrt{32.7519}$$

$$\|A\|_\infty = \max\{6, 2, 8\} = 8$$

Condition Number

$$
\begin{aligned}
\|A\|_F &= \sqrt{\text{trace}(A^T A)} = \sqrt{34} \\
\|A^{-1}\|_1 &= \max\{3/2, 9/2, 3/2\} = 9/2 \\
\|A^{-1}\|_2 &= \sqrt{9.34315} \\
\|A^{-1}\|_\infty &= \max\{5/2, 7/2, 3/2\} = 7/2 \\
\|A^{-1}\|_F &= \sqrt{\text{trace}(A^{-T} A^{-1})} = \sqrt{41/4} = 3.2016 \\
\kappa(A)_1 &= 27 \\
\kappa(A)_2 &= 17.4930 \\
\kappa(A)_\infty &= 28 \\
\kappa(A)_F &= 18.6682
\end{aligned}
$$

Condition Number

From this you can verify that

$$\|A_2\| \leq \|A\|_F \leq \sqrt{3}\|A\|_2$$

Is this true for any general n?

Challenges

Challenges

1. Definition of condition number involves matrix inverse, which is not easy to compute and requires more work than solving the system $Ax = b$.
2. In practice, condition number is merely estimated, using another relative expensive byproduct of the solution procedure.
3. If z is a solution of $Ax = y$, then

$$\|z\| = \|A^{-1}y\| \leq \|A^{-1}y\| \implies \frac{\|z\|}{\|y\|} \leq \|A^{-1}\|$$

4. If we can find a vector y such that $\frac{\|z\|}{\|y\|}$ is as large as possible, then we can find an reasonable estimate for $\|A^{-1}\|$.

Challenges

1. Usually, the ℓ_2 norm or spectral norm is used for the condition number as it provides the tightest measure of size.
2. Also, this norm does not require any matrix inverse, instead computation of eigenvalues. With the help of Gersghorin's theorem and Spectral radius, one can find this.
3. The condition number of a matrix in ℓ_2 norm can be defined as

$$\kappa(A) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$$

where σ_{max} and σ_{min} denote the maximal and minimal singular values of A respectively.

Challenges

1. If A is normal ($A^*A = AA^*$), then

$$\kappa(A) = \frac{|\lambda_{max}(A)|}{|\lambda_{min}(A)|}$$

2. If A is unitary $A^*A = AA^* = I$, then

$$\kappa(A) = 1$$

3. If $\|.\|$ is the ℓ_∞ norm and A is a nonsingular lower or upper triangular matrix, then

$$\kappa(A) \geq \frac{\max_{1 \leq i \leq n} |a_{ii}|}{\min_{1 \leq i \leq n} |a_{ii}|}$$

Properties

Verify whether the following Statements are True or Not.

1. For any matrix A, $\kappa(A) \geq 1$
2. $\kappa(I) = 1$
3. $\kappa(\alpha A) = \kappa(A)$
4. $\kappa(D) = \frac{\max_i |d_i|}{\min_i |d_i|}$
5. $\kappa(AB) \leq \kappa(A)\kappa(B)$
6. If $A = A^T$, then $\kappa(A^2) = \kappa(A)^2$

Theorems

Theorem 21

If A is and $n \times n$ matrix such that $\|A\| < 1$, then $I - A$ is invertible, and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

From this theorem, we can observe that

$$\|(I - A)^{-1}\| \leq \sum_{k=0}^{\infty} \|A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|}$$

Theorems

Theorem 22

If A and B are $n \times n$ matrices such that $\|I - AB\| < 1$, then A and B are invertible. Furthermore,

$$A^{-1} = B \sum_{k=0}^{\infty} (I - AB)^k$$

and

$$B^{-1} = \sum_{k=0}^{\infty} (I - AB)^k A$$

Theorems

Theorem 23

If A and B are $n \times n$ matrices such that $\|I - AB\| < 1$, then A and B are invertible. Furthermore,

$$A^{-1} = B \sum_{k=0}^{\infty} (I - AB)^k$$

and

$$B^{-1} = \sum_{k=0}^{\infty} (I - AB)^k A$$

Theorems

Proof: By previous theorem, $AB = I - (I - AB)$ is invertible and

$$(AB)^{-1} = \sum_{k=0}^{\infty} (I - AB)^k$$

$$A^{-1} = B(AB)^{-1} = B \sum_{k=0}^{\infty} (I - AB)^k$$

$$B^{-1} = (AB)^{-1} A = \sum_{k=0}^{\infty} (I - AB)^k A$$

Iterative Methods

Theorems

Theorem 24

If A is and $n \times n$ matrix such that $\|A\| < 1$, then $I - A$ is invertible, and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

From this theorem, we can observe that

$$\|(I - A)^{-1}\| \leq \sum_{k=0}^{\infty} \|A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|}$$

Theorems

Theorem 25

If A and B are $n \times n$ matrices such that $\|I - AB\| < 1$, then A and B are invertible. Furthermore,

$$A^{-1} = B \sum_{k=0}^{\infty} (I - AB)^k$$

and

$$B^{-1} = \sum_{k=0}^{\infty} (I - AB)^k A$$

Iterative Methods

The general algorithm for solving a system $Ax = b$ is as follows:

1. Choose a nonsingular matrix Q
2. Choose an arbitrary starting vector $x^{(0)}$
3. Generate vectors $x^{(1)}, x^{(2)}, \cdots$ recursively from the equation

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b, \quad k = 1, 2, 3, \cdots \tag{3}$$

Suppose x is the solution of the system $Ax = b$ and $x^{(k)}$ converges to x, as $k \to \infty$, then

$$Qx = (Q - A)x + b$$

Note, that the system (3) should be easy to solve for $x^{(k)}$ when the right hand side is known. Also, Q should be chosen to ensure that $x^{(k)}$ converges to x, no matter, what initial vector is used and the convergence should be rapid.

Iterative Methods

Note that the true solution x satisfies the equation

$$x = (I - Q^{-1}A)x + Q^{-1}b \tag{4}$$

Therefore, x is a fixed point of the mapping

$$f(x) = (I - Q^{-1}A)x + Q^{-1}b$$

From (3)

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b, \quad k = 1, 2, 3, \cdots$$

$$\implies x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b, \quad k = 1, 2, 3, \cdots$$

Now,

$$x^{(k)} - x = (I - Q^{-1}A)(x^{(k-1)} - x), \quad k = 1, 2, 3, \cdots \tag{5}$$

$$\implies \|x^{(k)} - x\| = \|(I - Q^{-1}A)\|\|(x^{(k-1)}) - x\|$$

$$\implies \|x^{(k)} - x\| = \|(I - Q^{-1}A)\|^{k}\|(x^{(0)}) - x\|$$

Iterative Methods

If $\|I - Q^{-1}A\| < 1$, we can conclude that

$$\lim_{k\to\infty} \|x^{(k)} - x\| = 0$$

By above theorem, it is guaranteed that if $\|I - Q^{-1}A\| < 1$, then both $Q^{-1}A$ and A are invertible.

Theorem 26

If $\|I - Q^{-1}A\| < 1$ for some matrix norm, then the sequence produced by (3) converges to the solution of $Ax = b$ for any initial vector $x^{(0)}$.

Theorem 27

If all eigenvalues of $I - Q^{-1}A$ lies in the open unit disc $|z| < 1$, then the sequence produced by (3) converges to the solution of $Ax = b$ for any initial vector $x^{(0)}$.

Iterative Methods

The above theorem implies that the spectral radius of $I - Q^{-1}A$ must be less than 1, that is,

$$\rho(I - Q^{-1}A) < 1$$

Let $r^{(k)}$ denote the residual vector obtained from $x^{(k)}$ after k iterations, then we get

$$r^{(k)} = b - Ax^{(k)}.$$

By above theorem, if $\|I - Q^{-1}A\| < 1$, then $\|r^{(k)}\| \to 0$.

Thanks

Doubts and Suggestions

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MA633L-Numerical Analysis

Lecture 27 : Numerical Linear Algebra - Matrix Norms

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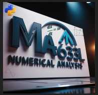

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