MA633L-Numerical Analysis

Lecture 31 : Numerical Linear Algebra - Eigenvalues

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Theorems on Eigenvalues

Theorems on Eigenvalues

Theorem 1

The following statements are true for any square matrix A

1. If λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue of $p(A)$ for any polynomial p. In particular λ^k is an eigenvalue of A^k.
2. If A is invertible and λ is an eigenvalue of A, then $p(1/\lambda)$ is an eigenvalue of $p(A^{-1})$ for any polynomial p. In particular λ^{-1} is an eigenvalue of A^{-1}.
3. If A is a real and symmetric, then its eigenvalues are real.
4. If A is complex and Hermitian, then its eigenvalues are real.
5. If A is Hermitian and positive definite, then its eigenvalues are positive.
6. If P is invertible, then A and PAP^{-1} have the same characteristic polynomial and the same eigenvalues.

Theorems on Eigenvalues

Theorem 2
Similar matrices have the same eigenvalues

Theorem 3 (Schur's Theorem)
Every square matrix is unitarily similar to a triangular matrix

Corollary 1
Every square real matrix is similar to a triangular matrix

Corollary 2
Every square Hermitian matrix is unitarily similar to a diagonal matrix

Few Applications of Eigenvalues

- PageRank Algorithm: Uses eigenvectors to rank web pages based on their importance. THE $25,000,000,000* EIGENVECTOR THE LINEAR ALGEBRA BEHIND GOOGLE
- Principal Component Analysis (PCA): Eigenpairs to find the direction of maximum variance in high-dimensional data Link
- Latent Semantic Analysis: Reduce the dimensionality of textual data in NLP Link.
- Face Recognition (Eigenfaces): Uses PCA to project facial images onto a lower dimensional eigenface space Link
- Compression algorithms: SVD

Power Method

Power Method

- This method is used to compute eigenvalues of a given matrix.
- Let A be an $n \times n$ matrix. Assume that eigenvalues of A satisfies the following property:

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

- Suppose each eigenvalue has a nonzero eigenvector $u^{(i)}$ and

$$Au^{(i)} = \lambda_i u^{(i)}, \quad i = 1, 2, \cdots, n$$

- Suppose $\{u^{(1)}, u^{(2)}, \cdots, u^{(n)}\}$ is a linearly independent set of n eigenvectors and it forms a basis for $\mathbb{C}^n$. (Do we really need linearly independent condition?)

Power Method

- Power method computes the dominant eigenvalue, that is λ_1.
- Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary starting vector.
- Since $\{u^{(1)}, u^{(2)}, \cdots, u^{(n)}\}$ is a basis for $C^{(n)}$, we have

$$x^{(0)} = \sum_{i=1}^{n} c_i u^{(i)}$$

- Assume that $c_1 \neq 0$. Further, we can absorb c_i into $u^{(i)}$ and write it as

$$x^{(0)} = \sum_{i=1}^{n} u^{(i)}$$

Power Method

- Let

$$x^{(1)} = Ax^{(0)}$$

  and

$$x^{(k)} = Ax^{(k-1)}$$

- It is easy to claim that

$$x^{(k)} = A^k x^{(0)}$$

Therefore,

$$x^{(k)} = A^k x^{(0)} = A^k \sum_{i=1}^{n} u^{(i)} = \sum_{i=1}^{n} \lambda_i^k u^{(i)}$$

Power Method

$$x^{(k)} = \lambda_1^k \left[u^{(1)} + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1} \right)^k u^{(i)} \right]$$

Since $|\lambda_1| > |\lambda_i|$ for $i > 1$, we have

$$\frac{\lambda_i}{\lambda_1} < 1$$

$$\lim_{k \to \infty} \left(\frac{\lambda_i}{\lambda_1} \right)^k = 0$$

Power Method

$$x^{(k)} = \lambda_1^k[u^{(1)} + \epsilon^{(k)}]$$

Here

$$\lim_{k\to\infty} \epsilon^{(k)} = 0$$

Now, let ϕ be any complex-valued linear functional on $\mathbb{C}^n$ such that

$$\phi(u^{(1)}) \neq 0$$

Note: A function ϕ is said to be linear functional if $\phi(ax + by) = a\phi(x) + b\phi(y)$ for scalars a and b and vectors x and y. **Example:** $\phi(x) = x_j$ is a linear functional.

Power Method

$$\phi(x^{(k)}) = \lambda_1^k[\phi(u^{(1)}) + \phi(\epsilon^{(k)})]$$

Define r_k as follows

$$r_k = \frac{\phi(x^{(k+1)})}{\phi(x^{(k)})} = \lambda_1 \left[\frac{\phi(u^{(1)}) + \phi(\epsilon^{(k+1)})}{\phi(u^{(1)}) + \phi(\epsilon^{(k)})} \right]$$

Then, it is easy to verify that

$$\lim_{k\to\infty} r_k = \lambda_1$$

This gives the dominant eigenvalue λ_1. In fact, we can obtain the respective eigenvector also after a careful manipulation.

Power Method

Example 4

Find the dominant eigenvalue of the following matrix

$$\begin{bmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{bmatrix}, x^{(0)} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \phi(x) = x_2$$

$$x^{(0)} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}, r_0 = 2$$

$$x^{(1)} = \begin{bmatrix} -0.90453 & 0.3015 & 0.30151 \end{bmatrix}, r_1 = -2$$

$$\vdots$$

$$x^{(31)} = \begin{bmatrix} -0.577352 & -0.577352 & -0.577352 \end{bmatrix}, r_{31} = 6.00001$$

Inverse Power Method

When A is invertible, it is possible to find the least eigenvalue using inverse power method. When λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Suppose λ_1 is the only smallest eigenvalue of A, then $\frac{1}{\lambda}$ is the dominant eigenvalue of A^{-1}. That is,

$$|\lambda_1^{-1}| > |\lambda_2^{-1}| \geq |\lambda_3^{-1}| \geq \cdots \geq |\lambda_n^{-1}| > 0$$

We can use power method on the matrix A^{-1}.

$$x^{(k+1)} = A^{-1}x^{(k)} \implies Ax^{(k+1)} = x^{(k)}$$

Now, we need to solve this system using either LU decomposition or other methods which we solved earlier.

Shifted Inverse Power Method

When we want to compute the eigenvalue of A that is close to a given number μ, we can apply the inverse power method to $A - \mu I$ and take the reciprocal of the limiting value of r, this should be $\lambda - \mu$. That is,

$$Ax = \lambda x \iff (A - \mu I)x = (\lambda - \mu)x \iff (A - \mu I)^{-1}x = \frac{1}{\lambda - \mu}x$$

Suppose that some eigenvalue λ_j of matrix A, we have $|\lambda_j - \mu| > \epsilon$ and $0 < |\lambda_i - \mu| < \epsilon$ for all $i \neq j$. Consider the shifted matrix $A - \mu I$, applying power method to the shifted matrix, $A - \mu I$, we compute ratios r_k that converge to $\lambda_j - \mu$. This procedure is called **shifted power method.** Similarly we can compute $\frac{1}{\lambda - \mu}$ and this algorithm is called **shifted inverse power method.**

Aitken Acceleration

Using the Aitken acceleration formula, the sequence $\{r_k\}$ can be accelerated to converge to r by Aitken acceleration.

$$s_k = r_k - \frac{(r_k - r_{k-1}^2)}{r_k - 2r_{k-1} + r_{k-2}}, k \geq 3$$

Other Eigenvalue Algorithm

Rayleigh Quotient

Definition 5 (Rayleigh Quotient)
The Rayleigh quotient is the expression

$$\frac{\langle x, x\rangle_A}{\langle x, x\rangle} = \frac{x^T A x}{x^T x}$$

Let λ_0 be the initial guess of an eigenvalue for the Hermitian matrix A and $x^{(0)}$ be the initial eigenvector guess, then

$$x^{(i+1)} = \frac{(A - \lambda_i I)^{-1} x^{(i)}}{\|(A - \lambda_i I)^{-1} x^{(i)}\|}$$

and

$$\lambda_{i+1} = \frac{x^{(i+1)*} A x^{(i+1)}}{x^{(i+1)*} x^{(i+1)}}$$

provides the eigenvector and eigenvalue.

QR Algorithm

For real matrix, if we want to compute the eigenvalues, we can employ $QR-$ decomposition. Assume $A_0 = A$. At k^{th} step obtain the QR decomposition

$$A_k = Q_k R_k$$

where Q_k is an orthogonal matrix and R_k is an upper triangular matrix. Then form

$$A_{k+1} = R_k Q_k$$

Note:

$$A_{k+1} = R_k Q_k = Q_k^{-1} Q_k R_k Q_k = Q_k^{-1} A_k Q_k = Q_k^T A_k Q_k$$

Therefore, A_{k+1} and A_k are similar matrices and have same eigenvalues.

QR Algorithm

When conditions are imposed on A, then the matrices A_k converge to a triangular matrix, the Shcur form of A. We know eigenvalue of a triangular matrix are its diagonal entries, therefore, we can obtain all eigenvalues of A.
Note: QR method is relatively expensive, it requires $O(n^3)$ operations for one step. The algorihm requires $O(n^4)$ operations.

Krylov Subspace methods

We can also employ Krylov subspace methods to compute eigenvalues such as

1. Arnoldi iteration
2. Lanczos algorithm
3. Block Lanczos algorithm

Other methods are

1. Divide and Conquer algorithm
2. Jacobi eigenvalue algorithm
3. Folded spectrum method
4. Locally optimal Block Preconditioned Conjugate Gradient Method.

Few Unsolved Problems in Numerical Linear Algebra

Open Problems in NLA

1. Hadamard Conjecture
2. Eigenvalue Distribution Conjecture for Random matrices
3. Necessary and sufficient conditions for a set of n real numbers to be the eigenvalues of a symmetric nonnegative matrix of order n
4. S-matrix conjecture
5. Jacobian Conjecture
6. Crouzeix’s conjecture
7. Non-negative inverse Eigenvalue Problems (NIEP)

Open Problems 1 Open Problems 2

Thanks

Doubts and Suggestions

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