MA633L-Numerical Analysis

Lecture 32 : Numerical Integration - Newton-Cotes

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

April 3, 2025

Numerical Integration

Numerical Integration

- In school days or in your calculus course, you have studied the integration.
- When a function in one variable is given and an interval is given, then integration (under certain conditions) represents the area under the curve.
- The dictionary meaning of integration is "to bring together as parts into a whole; to unite; to indicate the total amount".
- Mathematically it represents the summation. When the sum is in discrete sense we use the symbol $\sum$ whereas for continuum cases, we represent it with $\int$ where this symbol is elongated and stylised 'S'.

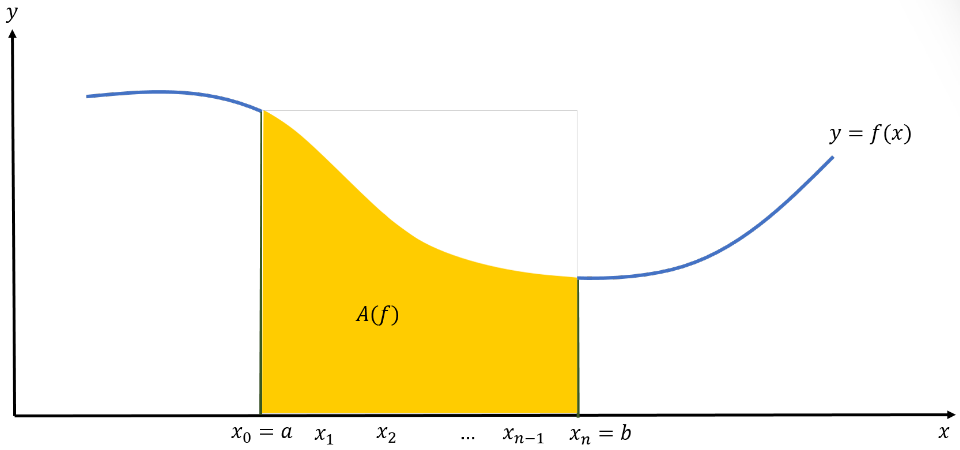
Numerical Integration

- The integration is represented by

$$I = \int_{a}^{b} f(x)dx \tag{1}$$

which represents the integral of the function $f(x)$ with respect to the independent variable x, evaluated between the limits $x = a$ to $x = b$. It represents the summation of $f(x)dx$ over the range $x = a$ to $x = b$. From below figure, the area under the curve $f(x)$ between $x = a$ and $x = b$ is given by $A(f)$.

Numerical Integration



Numerical Integration

From your calculus course, you could recall that, this can be calculated using the upper Riemann sum and lower Riemann sum where the former fills the area with larger rectangles and the latter with smaller rectangles.

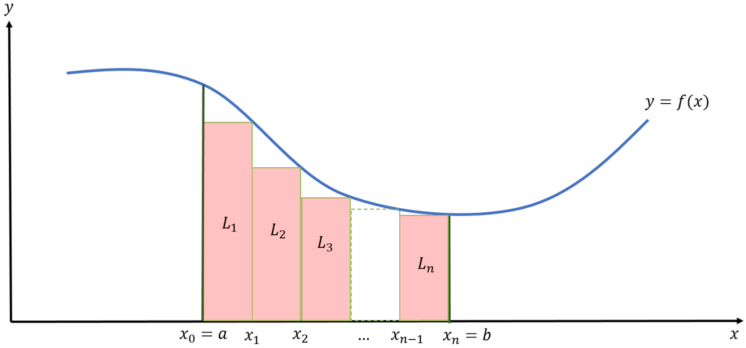


Figure 1: Lower Riemann Sum

Numerical Integration

Each ith rectangle takes the width as $x_i - x_{i-1}$ and height as respectively $M_i = \max_{x_{i-1} \le x \le x_i} f(x)$ and $m_i = \min_{x_{i-1} \le x \le x_i} f(x)$ for upper and lower Riemann rectangles. Then the area of each upper Riemann rectangles U_i is calculated by $U_i = M_i(x_i - x_{i-1})$ and the area of each lower Riemann rectangles L_i is calculated by $L_i = m_i(x_i - x_{i-1})$.

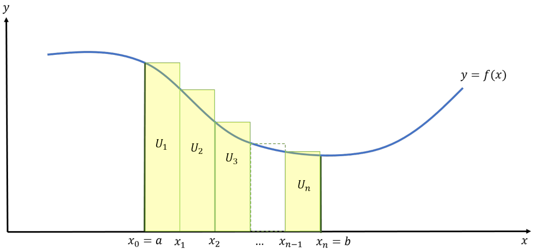


Figure 2: Upper Riemann Sum

Numerical Integration

- According to the Riemann sum, the area $A(f)$ is bounded between the total sum of all area of lower Riemann rectangles and upper Riemann integrals, that is,

$$\sum_{i=1}^{n} L_i \leq A(f) \leq \sum_{i=1}^{n} U_i$$

or

$$L(P_n, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq A(f) \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = U(P_n, f)$$

- When the partition is made finer and finer or $n \to \infty$, we obtain the integral as

$$A(f) = \int_a^b f(x)dx$$

Numerical Integration

- Numerical integration are often referred as quadrature meant to construct square having the same area as a curvilinear figure. However, it refers to numerical definite integration nowadays.
- Integration is widely used by engineers to compute the mass, center of gravity in civil and mechanical engineering, root mean square current in electrical engineering, total mass of a chemical present in a reactor when the concentration varies with respect to the location, energy transfer in food processing engineering and so on.

Numerical Integration

- In your real analysis or basic calculus course, you should have explored a variety of integration and computed them exactly, but they are only limited.
- In practice, it is not possible to compute the integration analytically.
- Also, when the measurements are known only at discrete points, it is not possible to obtain the integration. Numerical integration technique could approximate the values.
- Numerical integration is classified as Newton-Cotes formulas, Romberg Integration, Quadrature rules.

Numerical Integration-Newton-Cotes

Newton-Cotes

- Newton-Cotes formula is further divided into closed forms and open forms.
- In the closed forms, it considers the end points of the closed interval $[a, b]$, whereas open form does not.
- The idea is once again to use Weierstrass approximation theorem.
- From real analysis course, we know that if the function is continuous, it is integrable.
- By Weierstrass approximation theorem, each continuous function is approximated by a polynomial of degree n.
- Since integrating the polynomial is an easier job compare to an arbitrary continuous function $f(x)$, Newton-Cotes formula exploits it.

Newton-Cotes

- Suppose $f \in C[a,b]$, then there exists a polynomial $P_n(x)$ such that

$$|f - P_n| < \epsilon$$

- The corresponding integration is given by

$$I = \int_a^b f(x)dx \cong \int_a^b P_n(x)dx$$

- Instead of using a single polynomial, when different polynomials are used at each of the ith interval, it is termed as composite rule.

$$I = \int_a^b f(x)dx \cong \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} P_{n_i}(x)dx$$

- When a first order polynomial is applied we obtain trapezoidal rule, for second and third order polynomials respectively we obtain the Simpson's 1/3 and Simpsons 3/8 rules.

Numerical Integration-Basic Trapezoidal Rule

Basic Trapezoidal Rule

The trapezoidal rule is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial is $P_1(x)$. When $(a, f(a))$ and $(b, f(b))$ are two given points, then the line joining these two points are given by

$$\frac{y-(f(a)}{f(b)-f(a)} = \frac{x-a}{b-a}$$

$$y = f(a) + \frac{x-a}{b-a}(f(b)-f(a))$$

Therefore, the polynomial is given by

$$P_1(x) = f(a) + \frac{x-a}{b-a}(f(b)-f(a)) \tag{2}$$

Basic Trapezoidal Rule

Therefore, the integration is given by

$$
\begin{aligned}
I = \int_a^b f(x)dx &= \int_a^b P_1(x)dx \\
&= \int_a^b f(a) + \frac{x-a}{b-a}(f(b) - f(a))dx \\
&= f(a)[x]_a^b + \left[\frac{(x-a)^2}{2(b-a)}(f(b) - f(a))\right]_a^b \\
&= f(a)(b-a) + \frac{1}{2}[f(b) - f(a)](b-a) \\
&= (b-a)\left[f(a) + \frac{f(b)}{2} - \frac{f(a)}{2}\right] \\
&= (b-a)\frac{f(a)+f(b)}{2}
\end{aligned}
$$

Basic Trapezoidal Rule

The equation

$$I = (b-a)\frac{f(a)+f(b)}{2} \tag{3}$$

is called the trapezoidal rule. It is equivalent to approximating the area of the trapezoid under the line joining $(a, f(a))$ and $(b, f(b))$. From school mathematics, we know that finding the area of trapezoid is the product of the height and the average of the bases, here the bases are $f(a)$ and $f(b)$ and the height is $(b-a)$, that is

$$I = \text{Width} \times \text{ Average Height}$$

Basic Trapezoidal Rule

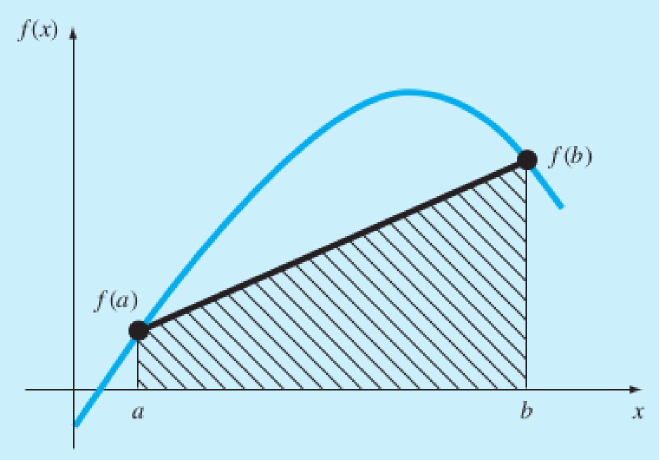


Figure 3: trapezoidal Rule

Basic Trapezoidal Rule

All Newton-cotes closed formula can be expressed in the above format with different average heights. If we represent $(x_0, f(x_0)) = (a, f(a))$ and $(x_1, f(x_1)) = (b, f(b))$, then we can use Lagrange polynomial

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

and the trapezoidal rule becomes

$$I = (x_1 - x_0) \frac{f(x_0) + f(x_1)}{2} \tag{4}$$

Error in Trapezoidal Rule

Error in Trapezoidal Rule

Theorem 1 (Error in trapezoidal Rule)
If $f \in C^2[a,b]$, then the error in trapezoidal rule is given by

$$-\frac{(b-a)^3}{12} f''(\xi)$$

for some $\xi \in (a,b)$

Proof:
The proof is again using the first interpolation error theorem,

$$f(x) - P_1(x) = \frac{1}{2!} f''(\xi)(x-a)(x-b)$$

$$\implies \int_a^b (f(x) - P_1(x))dx = \frac{1}{2!} f''(\xi) \int_a^b (x-a)(x-b)dx$$

Error in Trapezoidal Rule

Now

$$
\begin{aligned}
\int_a^b (x-a)(x-b)dx &= \frac{1}{2!}\int_a^b (x-a)d\frac{(x-b)^2}{2} \\
&= \left[(x-a)\frac{(x-b)^2}{2}\right]_a^b - \int_a^b \frac{(x-b)^2}{2}dx \\
&= -\left[\frac{(x-b)^3}{6}\right]_a^b = \frac{(a-b)^3}{6} = -\frac{(b-a)^3}{6}
\end{aligned}
$$

$$
\implies \int_a^b (f(x) - P_1(x))dx = \frac{1}{2!}f''(\xi)\int_a^b (x-a)(x-b)dx = -\frac{(b-a)^3}{12}f''(\xi)
$$

Examples: Trapezoidal Rule

Examples: Trapezoidal Rule

Example 2

Evaluate

$$\int_0^1 e^{-x^2}\,dx$$

using trapezoidal rule.

$$a = 0, b = 1, f(a) = 1, f(b) = 0.36878$$
$$I = (b-a)\frac{f(a)+f(b)}{2} = 0.6839$$

Examples: Trapezoidal Rule

Example 3

$$
\begin{aligned}
\int_0^2 x^2 dx &= (2-0)\frac{0+4}{2} = 4 \\
\int_0^2 x^4 dx &= (2-0)\frac{0+2^4}{2} = 16 \\
\int_0^2 \frac{1}{1+x} dx &= (2-0)\frac{1+0.3333}{2} = 1.333 \\
\int_0^2 \sqrt{1+x^2} dx &= (2-0)\frac{1+\sqrt{5}}{2} = 3.236 \\
\int_0^2 e^x dx &= (2-0)\frac{1+e^2}{2} = 8.389
\end{aligned}
$$

Closed Newton-Cotes Formula

Closed Newton-Cotes Formula

The general $(n+1)$-points closed Newton-Cotes formula using nodes $x_i = x_0 + ih$, for $i = 0, 1, 2, \cdots n$ where $x_0 = a, x_n = b, h = (b-a)/n$ is

$$\int_a^b f(x)dx \cong \sum_{i=0}^{n} \left[\int_{x_0}^{x_n} \ell_i(x)dx \right] f(x_i) \tag{5}$$

where ℓ_i's are cardinal polynomials as we discussed in Lagrange Interpolation. This method is referred as closed as it includes $x_0 = a$ and $x_n = b$ in its computation. Equation (5) can also be written as

$$\int_a^b f(x)dx \cong \sum_{i=0}^{n} A_i f(x_i) \tag{6}$$

Method of undetermined coefficients

- In fact, these values of $A's$ can be obtained in a different way.
- Since f is continuous and we could approximate them using polynomials, the following tricks could help us derive the equation in an easy way.
- Since the basis of any polynomial of degree n is $\{1, x, x^2, \cdots, x^n\}$, which has $n+1$ elements, we can obtain the values of A_i by evaluating it for these basis functions.
- This process is called **method of undetermined coefficients.**

Method of undetermined coefficients

- Also, for simplifications purpose, let us use the change of variables and following assumptions.
- **Change of intervals:** Let us formulate the rules usually on an interval $[0,1]$ or $[-1,1]$ and transform to any $[c,d]$ using change of intervals.
- If the formula obtained from (2) is exact for any polynomial of certain degree over the first interval, the same is true for the transformed interval by below theorem.

Closed Newton-Cotes Formula

In general to transform the interval $[a, b]$ to $[c, d]$, we can use the following linear map, $\gamma : [a, b] \to [c, d]$ defined by (Derive!)

$$\gamma(x) = \left(\frac{d-c}{b-a}\right) x + \frac{bc - ad}{b-a}$$

Theorem 4

If γ' is continuous on the interval $[a, b]$ and f is continuous on the range $\gamma(x) = u$, then

$$\int_a^b f(\gamma(x)).\gamma'(x)dx = \int_{\gamma(a)}^{\gamma(b)} f(u)du \tag{7}$$

Closed Newton-Cotes Formula

By applying this theorem for our linear map γ which is differentiable and γ' is continuous we obtain that

$$\gamma'(x) = \frac{d-c}{b-a}$$
$$\gamma(a) = c$$
$$\gamma(b) = d$$
$$\int_a^b f(\gamma(x))\frac{d-c}{b-a}dx = \int_c^d f(u)du$$

Closed Newton-Cotes Formula

Similarly, if we define $\gamma^{-1} = \lambda : [c, d] \to [a, b]$ as

$$\lambda(x) = \left(\frac{b-a}{d-c}\right) x + \frac{ad-bc}{d-c}$$

then it can be written

$$\begin{aligned}\int_a^b f(x)dx &= \int_c^d f(\lambda(u))\lambda'(u)du \\ &= \frac{b-a}{d-c}\int_c^d f(\lambda(u))du\end{aligned}$$

Closed Newton-Cotes Formula

Now, if define $\lambda : [0, 1] \to [a, b]$ as $\lambda(x) = (b - a)x + a$, then $\lambda(0) = a, \lambda(1) = b$. Therefore, it becomes

$$\int_a^b f(x)dx = (b - a) \int_0^1 f(\lambda(u))du \tag{8}$$

Now, if define $\lambda : [-1, 1] \to [a, b]$ as

$$\lambda(x) = \frac{1}{2}(b - a)x + \frac{1}{2}(a + b),$$

then $\lambda(-1) = a, \lambda(0) = \frac{a+b}{2}, \lambda(1) = b$. Therefore, it becomes

$$\int_a^b f(x)dx = \frac{1}{2}(b - a) \int_{-1}^1 f(\lambda(u))du \tag{9}$$

Trapezoidal Rule

Let us obtain the basic trapezoidal rule using this process. Now, for trapezoidal rule, we obtain the formula on the interval $[0, 1]$. Trapezoidal rule uses a linear polynomial, therefore, we need to identify two unknowns A_0, A_1 The Newton-Cotes formula becomes

$$\int_0^1 f(x)dx \cong A_0 f(0) + A_1 f(1) \tag{10}$$

Trapezoidal Rule

By the method of undetermined coefficients, we evaluate them on $1, x$.

$$f(x) = 1 : \int_0^1 dx = A_0 + A_1 \implies A_0 + A_1 = 1$$

$$f(x) = x : \int_0^1 x dx = A_1 \implies A_1 = \frac{1}{2}$$

Upon simplification, we obtain that

$$A_0 = A_1 = \frac{1}{2}$$

Trapezoidal Rule

The resulting formula is

$$\int_0^1 f(x)dx \approx \frac{1}{2}[f(0) + f(1)] \tag{11}$$

By (8), we obtain that

$$\begin{aligned}\int_a^b f(x)dx = (b-a)\int_0^1 f(\lambda(u))du &\cong (b-a)[A_0 f(\lambda(0)) + A_1 f(\lambda(1))] \\ &\approx \frac{1}{2}(b-a)[f(a) + f(b)]\end{aligned}$$

If $h = (b-a)$, then

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a) + f(b)]$$

Simpson's 1/3 Rule

For Simpson's 1/3 rule, we use a quadratic polynomial, therefore, we need to identify three unknowns A_0, A_1, A_2. To obtain A_0, A_1, A_2, let us use $[-1, 1]$ interval. The Newton-Cotes formula becomes

$$\int_{-1}^{1} f(x)dx \cong A_0 f(-1) + A_1 f(0) + A_2 f(1) \tag{12}$$

$$f(x) = 1 : \int_{-1}^{1} dx = A_0 + A_1 + A_2 \implies A_0 + A_1 + A_2 = 2$$

$$f(x) = x : \int_{-1}^{1} xdx = -A_0 + A_2 \implies -A_0 + A_2 = 0$$

$$f(x) = x^2 : \int_{-1}^{1} x^2dx = A_0 + A_2 \implies A_0 + A_2 = \frac{2}{3}$$

Simpson's 1/3 Rule

Upon simplification, we obtain that

$$A_0 = \frac{1}{3}, A_2 = \frac{1}{3}, A_1 = \frac{4}{3}$$

The resulting formula is

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{3}[f(-1) + 4f(0) + f(1)] \tag{13}$$

Simpson's 1/3 Rule

By (9), we obtain that

$$
\begin{aligned}
\int_a^b f(x)dx &= \frac{1}{2}(b-a)\int_{-1}^{1} f(\lambda(u))du \\
&\cong \frac{1}{2}(b-a)]A_0 f(\lambda(-1)) + A_1 f(\lambda(0)) + A_2 f(\lambda(1))] \\
&\approx \frac{1}{6}(b-a)[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]
\end{aligned}
$$

If $h = (b-a)/2$, then

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(b)]$$

Simpson's 1/3 Rule

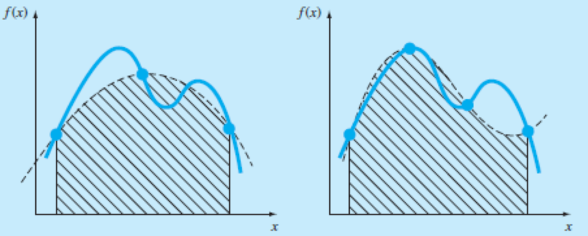


Figure 4: Simpson's 1/3 and 3/8 Rule

Simpson's 3/8 Rule

For Simpson's 3/8 rule, we obtain the formula on the interval $[0, 1]$. It uses a cubic polynomial, therefore, we need to identify four unknowns A_0, A_1, A_2, A_3 The Newton-Cotes formula becomes

$$\int_0^1 f(x)dx \cong A_0 f(0) + A_1 f\left(\frac{1}{3}\right) + A_2 f\left(\frac{2}{3}\right) + A_3 f(1) \tag{14}$$

Simpson's 3/8 Rule

By the method of undetermined coefficients, we evaluate them on $1, x, x^2, x^3$.

$$f(x) = 1 : \int_0^1 dx = A_0 + A_1 + A_2 + A_3 \implies A_0 + A_1 + A_2 + A_3 = 1$$

$$f(x) = x : \int_0^1 x dx = \frac{1}{3}A_1 + \frac{2}{3}A_2 + A_3 \implies \frac{1}{3}A_1 + \frac{2}{3}A_2 + A_3 = \frac{1}{2}$$

$$f(x) = x^2 : \int_0^1 x^2 dx = \frac{1}{9}A_1 + \frac{4}{9}A_2 + A_3 \implies \frac{1}{9}A_1 + \frac{4}{9}A_2 + A_3 = \frac{1}{3}$$

$$f(x) = x^3 : \int_0^1 x^3 dx = \frac{1}{27}A_1 + \frac{8}{27}A_2 + A_3 \implies \frac{1}{27}A_1 + \frac{8}{27}A_2 + A_3 = \frac{1}{4}$$

Simpson's 3/8 Rule

Upon simplification, we obtain that

$$A_0 = A_3 = \frac{1}{8}, A_1 = A_2 = \frac{3}{8}$$

The resulting formula is

$$\int_0^1 f(x)dx \approx \frac{1}{8}[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)] \tag{15}$$

Simpson's 3/8 Rule

By (8), we obtain that

$$\begin{aligned}\int_a^b f(x)dx &= (b-a)\int_0^1 f(\lambda(u))du \\ &\cong (b-a)\left[A_0 f(\lambda(0)) + A_1 f\left(\lambda\left(\frac{1}{3}\right)\right) + A_2 f\left(\lambda\left(\frac{2}{3}\right)\right) + A_3\lambda(1)\right] \\ &\approx \frac{1}{8}(b-a)\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]\end{aligned}$$

If $h = (b-a)/3$, then

$$\int_a^b f(x)dx \approx \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)]$$

Simpson's 3/8 Rule

Due to the fact that h is multiplied by $3/8$, it is referred as Simpson's 3/8 rule.

Theorem 5

Error in Simpson's 3/8 Rule

If $f \in C^4[a,b]$, then the error in Simpson's 3/8 rule is given by

$$-\frac{3}{80}\left(\frac{b-a}{3}\right)^5 f^{(4)}(\xi)$$

for some $\xi \in (a,b)$

Some Closed Newton-Cotes Formula

Some Closed Newton-Cotes Formula

When $n = 1$, $(n+1)$-points closed Newton-Cotes formula provides trapezoidal rule. Similarly Simpson's 1/3 rule and 3/8 rules are obtained respectively while choosing $n = 2$ and $n = 3$. When $n = 4$, we obtain the following Boole's rule

$$\int_a^b f(x)dx \approx \frac{2h}{45}[7f(a) + 32f(a+h) + 12f(a+2h) + 32f(a+3h) + 7f(b)]$$

with error as

$$E_4(t) = -\frac{8}{945}h^7 f^{(6)}(\xi), \xi \in (a, b)$$

Some Closed Newton-Cotes Formula

When $n = 5$, we obtain the six-point Newton-Cotes closed rule as follows:

$$\int_a^b f(x)dx \approx \frac{5h}{28}[19f(a) + 75f(a+h) + 50f(a+2h) + 50f(a+3h) \\ + 75f(a+4h) + 19f(b)]$$

where the error is

$$E_5(t) = -\frac{275}{12096}h^7 f^{(6)}(\xi), \xi \in (a, b)$$

The following theorem gives the general error term for an $(n+1)$-points closed Newton-Cotes formula. From this theorem, you can notice that when n is even, we obtain higher accuracy compared to odd. Therefore, in practice, h is considered as $h = (b-a)/(2^k)$, where $n = 2^k$.

Some Closed Newton-Cotes Formula

Theorem 6

Suppose that (5) denotes the $(n+1)-$ points closed Newton-Cotes formula with $x_0 = a, x_n = b$ and $h = (b-a)/n$. There exists $\xi \in (a,b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} A_i f(x_i) + E_n(t)$$

where the error is given by

$$E_n(t) = \begin{cases} \dfrac{h^{n+3}}{(n+2)!} f^{(n+2)}(\xi) \displaystyle\int_0^n t^2(t-1)\cdots(t-n)dt & n \text{ is even, } f \in C^{n+2}[a,b] \\ \dfrac{h^{n+2}}{(n+1)!} f^{(n+1)}(\xi) \displaystyle\int_0^n t(t-1)\cdots(t-n)dt & n \text{ is odd, } f \in C^{n+1}[a,b] \end{cases}$$

Open Newton-Cotes Formula

Open Newton-Cotes Formula

- The open Newton-Cotes formula do not include the endpoints $[a, b]$ as nodes.
- They use only interior nodes.
- The numerical approximation for open Newton-Cotes formula is same as (5), but, $x_i \in (a, b)$.
- In order to use the same formula (5), we redefine the points by spacing as follows: $x_{-1} = a, x_0 = a + h, x_{n+1} = b, x_i = x_0 + ih$ for each $i = 0, 1, 2, \cdots, n$ where $h = (b - a)/(n + 2)$.

Midpoint Rule

For the midpoint formula, we take $n = 0$, then $x_{-1} = a, x_0 = (a+b)/2, x_1 = b$. Therefore, the interior point is only $x_0 = (a+b)/2$. When we work on the interval $[0, 1]$, change of intervals and method of undetermined coefficients, we obtain that

$$\int_0^1 f(x)dx \cong A_0 f\left(\frac{1}{2}\right) \tag{16}$$

By the method of undetermined coefficients, we evaluate them on 1.

$$f(x) = 1 : \int_0^1 dx = A_0 \implies A_0 = 1$$

Midpoint Rule

The resulting formula is

$$\int_0^1 f(x)dx \approx f\left(\frac{1}{2}\right) \tag{17}$$

By (8), we obtain that

$$\begin{aligned}\int_a^b f(x)dx = (b-a)\int_0^1 f(\lambda(u))du &\cong (b-a)A_0 f\left(\lambda\left(\frac{1}{2}\right)\right)\\ &\approx (b-a)f\left(\frac{a+b}{2}\right)\end{aligned}$$

If $h = (b-a)/2$, then

$$\int_a^b f(x)dx \approx 2hf(a+h)$$

Midpoint Rule

Theorem 7 (Error in Midpoint Rule)
If $f \in C^2[a,b]$, then the error in Midpoint rule is given by

$$\frac{(b-a)^3}{24} f''(\xi)$$

Proof: By Taylor polynomial about $(a+b)/2$ for each $x \in (a,b)$ is given by

$$f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{1}{2} f''(\xi)\left(x - \frac{a+b}{2}\right)^2$$

Midpoint Rule

Integrating on both sides we obtain that

$$
\begin{aligned}
\int_a^b f(x)dx &= f\left(\frac{a+b}{2}\right)(b-a) + \frac{1}{2}f'\left(\frac{a+b}{2}\right)\underbrace{\left[\left(b-\frac{a+b}{2}\right)^2 - \left(a-\frac{a+b}{2}\right)^2\right]}_{0} \\
&\quad + \frac{1}{6}f''(\xi)\left[\left(b-\frac{a+b}{2}\right)^3 - \left(a-\frac{a+b}{2}\right)^3\right] \\
&= f\left(\frac{a+b}{2}\right)(b-a) + \frac{1}{3}f''(\xi)\left(\frac{b-a}{2}\right)^3
\end{aligned}
$$

Hence the proof.

Two-Point Newton-Cotes Open Rule

For the Two-point formula, we take $n = 1$, then $x_{-1} = a, x_0 = (a+b)/3, x_0 = 2(a+b)/3, x_2 = b$. Therefore, the interior points are only $x_0 = (a+b)/3$ and $x_1 = 2(a+b)/3$. When we work on the interval $[0,1]$, change of intervals and method of undetermined coefficients, we obtain that

$$\int_0^1 f(x)dx \cong A_0 f\left(\frac{1}{3}\right) + A_1 f\left(\frac{2}{3}\right) \tag{18}$$

Two-Point Newton-Cotes Open Rule

By the method of undetermined coefficients, we evaluate them on $\{1, x\}$.

$$f(x) = 1 : \int_0^1 dx = A_0 + A_1 \implies A_0 + A_1 = 1$$
$$f(x) = x : \int_0^1 x dx = A_0 \frac{1}{3} + A_1 \frac{2}{3} \implies \frac{1}{3} A_0 + \frac{2}{3} A_1 = \frac{1}{2}$$

Solving for A_0, A_1, we obtain that $A_0 = 1/2, A_1 = 1/2$.

Two-Point Newton-Cotes Open Rule

The resulting formula is

$$\int_0^1 f(x)dx \approx \frac{1}{2} f\left(\frac{1}{3}\right) + \frac{1}{2} f\left(\frac{2}{3}\right) \tag{19}$$

By (8), we obtain that

$$\begin{aligned}
\int_a^b f(x)dx &= (b-a)\int_0^1 f(\lambda(u))du \\
&\cong (b-a)A_0 f\left(\lambda\left(\frac{1}{3}\right)\right) + (b-a)A_1 f\left(\lambda\left(\frac{2}{3}\right)\right) \\
&\approx \frac{(b-a)}{2}\left[f\left(a+\frac{b-a}{3}\right) + f\left(a + 2\frac{b-a}{3}\right)\right]
\end{aligned}$$

Two-Point Newton-Cotes Open Rule

If $h = (b-a)/3$, then

$$\int_a^b f(x)dx \approx \frac{3h}{2}[f(a+h) + f(a+2h)]$$

Two-Point Newton-Cotes Open Rule

Theorem 8 (Error in Two-point Newton-Cotes Open Rule)
If $f \in C^2[a,b]$, then the error in Two-point rule is given by

$$\frac{(b-a)^3}{36} f''(\xi)$$

for some $\xi \in (a,b)$

Midpoint Rule

Example 9

Evaluate

$$\int_0^1 e^{-x^2} dx$$

using Midpoint rule.

$$a = 0, b = 1, \frac{a+b}{2} = 0.5, f\left(\frac{a+b}{2}\right) = 0.7788$$

$$\int_0^1 e^{-x^2} dx = (1-0)f(0.5) = 0.7788$$

Two-Point Newton-Cotes Open Rule

Example 10

Using Two-Point Newton-Cotes open rule, the following numerical integration can be obtained.

$$
\begin{aligned}
\int_0^2 x^2 dx &= \tfrac{3-0}{2}\left[\tfrac{4}{9} + \tfrac{16}{9}\right] = \tfrac{10}{3}, & E_t = \frac{2}{3} \\
\int_0^3 \frac{1}{1+x} dx &= \tfrac{3-0}{2}\left[0.5 + \tfrac{1}{3}\right] = 1.25, & E_t = 0.1363 \\
\int_0^3 \sqrt{1+x^2} dx &= \tfrac{3-0}{2}\left[\sqrt{2} + \sqrt{5}\right] = 5.0673, & E_t = -0.5853 \\
\int_0^3 e^x dx &= \tfrac{3-0}{2}\left[e + e^2\right] = 15.16, & E_t = 3.9246
\end{aligned}
$$

Some Open Newton-Cotes Formula

Some Open Newton-Cotes Formula

When $n = 0$, open Newton-Cotes formula provides Midpoint rule. Similarly, when $n = 1$, it yields the two-point Newton-Cotes open rule. When $n = 3$, the following three point Newton-Cotes open rule is obtained

$$\int_a^b f(x)dx \approx \frac{4h}{3}[2f(a+h) - f(a+2h) + 2f(a+3h)]$$

with error as

$$E_3(t) = \frac{14}{45}h^5 f^{(4)}(\xi), \xi \in (a,b)$$

Some Open Newton-Cotes Formula

When $n = 4$, we obtain the following four-point Newton-Cotes open rule

$$\int_a^b f(x)dx \approx \frac{5h}{24}[11f(a+h) + f(a+2h) + f(a+3h) + 11f(a+4h)]$$

where the error is

$$E_4(t) = \frac{95}{144}h^5 f^{(4)}(\xi), \xi \in (a,b)$$

Some Open Newton-Cotes Formula

When $n = 5$, we obtain the following five-point Newton-Cotes open rule

$$\int_a^b f(x)dx \approx \frac{6h}{20}[11f(a+h) - 14f(a+2h) + 26f(a+3h) - 14f(a+4h) + 11f(a+5h)]$$

where the error is

$$E_5(t) = -\frac{41}{140}h^7 f^{(6)}(\xi), \xi \in (a,b)$$

The following theorem gives the general error term for an $(n+1)$-points closed Newton-Cotes formula.

Some Open Newton-Cotes Formula

Theorem 11

Suppose that (5) denotes the open Newton-Cotes formula with $x_{-1} = a, x_{n+1} = b, x_0 = a+h, x_i = x_0 + ih, i = 0, 1, 2, \cdots, n$ where $h = (b-a)/(n+2)$. There exists $\xi \in (a, b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} A_i f(x_i) + E_n(t)$$

where the error is given by

$$E_n(t) = \begin{cases} \dfrac{h^{n+3}}{(n+2)!} f^{(n+2)}(\xi) \displaystyle\int_{-1}^{n+1} t^2(t-1)\cdots(t-n)dt & n \text{ is even, } f \in C^{n+2} \\ \dfrac{h^{n+2}}{(n+1)!} f^{(n+1)}(\xi) \displaystyle\int_{-1}^{n+1} t(t-1)\cdots(t-n)dt & n \text{ is odd, } f \in C^{n+1} \end{cases}$$

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in

MA633L-Numerical Analysis

Lecture 32 : Numerical Integration - Newton-Cotes

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

April 3, 2025

भारतीय प्रौद्योगिकी संस्थान तिरुपति
TIRUPATI