MA633L-Numerical Analysis

Lecture 32 : Numerical Integration - Newton-Cotes

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

April 4, 2025

Closed Newton-Cotes Formula

Closed Newton-Cotes Formula

The general $(n+1)$-points closed Newton-Cotes formula using nodes $x_i = x_0 + ih$, for $i = 0, 1, 2, \cdots n$ where $x_0 = a, x_n = b, h = (b-a)/n$ is

$$\int_a^b f(x)dx \cong \sum_{i=0}^{n} \left[\int_{x_0}^{x_n} \ell_i(x)dx \right] f(x_i) \tag{1}$$

where ℓ_i's are cardinal polynomials as we discussed in Lagrange Interpolation. This method is referred as closed as it includes $x_0 = a$ and $x_n = b$ in its computation. Equation (1) can also be written as

$$\int_a^b f(x)dx \cong \sum_{i=0}^{n} A_i f(x_i) \tag{2}$$

Method of undetermined coefficients

- In fact, these values of $A's$ can be obtained in a different way.
- Since f is continuous and we could approximate them using polynomials, the following tricks could help us derive the equation in an easy way.
- Since the basis of any polynomial of degree n is $\{1, x, x^2, \cdots, x^n\}$, which has $n+1$ elements, we can obtain the values of A_i by evaluating it for these basis functions.
- This process is called **method of undetermined coefficients.**

Method of undetermined coefficients

- Also, for simplifications purpose, let us use the change of variables and following assumptions.
- **Change of intervals:** Let us formulate the rules usually on an interval $[0, 1]$ or $[-1, 1]$ and transform to any $[c, d]$ using change of intervals.
- If the formula obtained from (2) is exact for any polynomial of certain degree over the first interval, the same is true for the transformed interval by below theorem.

Closed Newton-Cotes Formula

In general to transform the interval $[a, b]$ to $[c, d]$, we can use the following linear map, $\gamma : [a, b] \to [c, d]$ defined by (Derive!)

$$\gamma(x) = \left(\frac{d-c}{b-a}\right) x + \frac{bc - ad}{b-a}$$

Theorem 1

If γ' is continuous on the interval $[a, b]$ and f is continuous on the range $\gamma(x) = u$, then

$$\int_a^b f(\gamma(x)).\gamma'(x)dx = \int_{\gamma(a)}^{\gamma(b)} f(u)du \tag{3}$$

Closed Newton-Cotes Formula

By applying this theorem for our linear map γ which is differentiable and γ' is continuous we obtain that

$$\gamma'(x) = \frac{d-c}{b-a}$$
$$\gamma(a) = c$$
$$\gamma(b) = d$$
$$\int_a^b f(\gamma(x))\frac{d-c}{b-a}dx = \int_c^d f(u)du$$

Closed Newton-Cotes Formula

Similarly, if we define $\gamma^{-1} = \lambda : [c, d] \to [a, b]$ as

$$\lambda(x) = \left(\frac{b-a}{d-c}\right) x + \frac{ad - bc}{d - c}$$

then it can be written

$$\begin{aligned} \int_a^b f(x)dx &= \int_c^d f(\lambda(u))\lambda'(u)du \\ &= \frac{b-a}{d-c}\int_c^d f(\lambda(u))du \end{aligned}$$

Closed Newton-Cotes Formula

Now, if define $\lambda : [0,1] \to [a,b]$ as $\lambda(x) = (b-a)x + a$, then $\lambda(0) = a, \lambda(1) = b$. Therefore, it becomes

$$\int_a^b f(x)dx = (b-a)\int_0^1 f(\lambda(u))du \tag{4}$$

Now, if define $\lambda : [-1,1] \to [a,b]$ as

$$\lambda(x) = \frac{1}{2}(b-a)x + \frac{1}{2}(a+b),$$

then $\lambda(-1) = a, \lambda(0) = \frac{a+b}{2}, \lambda(1) = b$. Therefore, it becomes

$$\int_a^b f(x)dx = \frac{1}{2}(b-a)\int_{-1}^1 f(\lambda(u))du \tag{5}$$

Simpson's 1/3 Rule

For Simpson's 1/3 rule, we use a quadratic polynomial, therefore, we need to identify three unknowns A_0, A_1, A_2. To obtain A_0, A_1, A_2, let us use $[-1, 1]$ interval. The Newton-Cotes formula becomes

$$\int_{-1}^{1} f(x)dx \cong A_0 f(-1) + A_1 f(0) + A_2 f(1) \tag{6}$$

$$f(x) = 1 : \int_{-1}^{1} dx = A_0 + A_1 + A_2 \implies A_0 + A_1 + A_2 = 2$$

$$f(x) = x : \int_{-1}^{1} xdx = -A_0 + A_2 \implies -A_0 + A_2 = 0$$

$$f(x) = x^2 : \int_{-1}^{1} x^2 dx = A_0 + A_2 \implies A_0 + A_2 = \frac{2}{3}$$

Simpson’s 1/3 Rule

Upon simplification, we obtain that

$$A_0 = \frac{1}{3}, A_2 = \frac{1}{3}, A_1 = \frac{4}{3}$$

The resulting formula is

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{3}[f(-1) + 4f(0) + f(1)] \tag{7}$$

Simpson's 1/3 Rule

By (5), we obtain that

$$\begin{aligned}\int_a^b f(x)dx &= \frac{1}{2}(b-a)\int_{-1}^{1} f(\lambda(u))du \\ &\cong \frac{1}{2}(b-a)]A_0 f(\lambda(-1)) + A_1 f(\lambda(0)) + A_2 f(\lambda(1))] \\ &\approx \frac{1}{6}(b-a)[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]\end{aligned}$$

If $h = (b-a)/2$, then

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(b)]$$

Simpson’s 1/3 Rule

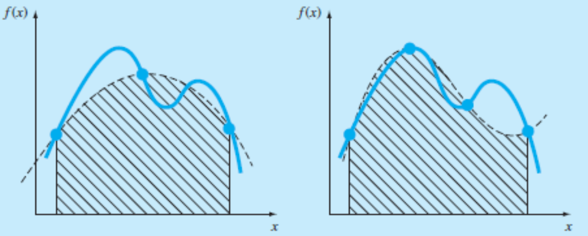

Figure 1: Simpson’s 1/3 and 3/8 Rule

Simpson's 3/8 Rule

For Simpson's 3/8 rule, we obtain the formula on the interval $[0, 1]$. It uses a cubic polynomial, therefore, we need to identify four unknowns A_0, A_1, A_2, A_3 The Newton-Cotes formula becomes

$$\int_0^1 f(x)dx \cong A_0 f(0) + A_1 f\left(\frac{1}{3}\right) + A_2 f\left(\frac{2}{3}\right) + A_3 f(1) \qquad (8)$$

Simpson's 3/8 Rule

By the method of undetermined coefficients, we evaluate them on $1, x, x^2, x^3$.

$$f(x) = 1 : \int_0^1 dx = A_0 + A_1 + A_2 + A_3 \implies A_0 + A_1 + A_2 + A_3 = 1$$

$$f(x) = x : \int_0^1 x dx = \frac{1}{3}A_1 + \frac{2}{3}A_2 + A_3 \implies \frac{1}{3}A_1 + \frac{2}{3}A_2 + A_3 = \frac{1}{2}$$

$$f(x) = x^2 : \int_0^1 x^2 dx = \frac{1}{9}A_1 + \frac{4}{9}A_2 + A_3 \implies \frac{1}{9}A_1 + \frac{4}{9}A_2 + A_3 = \frac{1}{3}$$

$$f(x) = x^3 : \int_0^1 x^3 dx = \frac{1}{27}A_1 + \frac{8}{27}A_2 + A_3 \implies \frac{1}{27}A_1 + \frac{8}{27}A_2 + A_3 = \frac{1}{4}$$

Simpson's 3/8 Rule

Upon simplification, we obtain that

$$A_0 = A_3 = \frac{1}{8}, A_1 = A_2 = \frac{3}{8}$$

The resulting formula is

$$\int_0^1 f(x)dx \approx \frac{1}{8}[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)] \tag{9}$$

Simpson's 3/8 Rule

By (4), we obtain that

$$
\begin{aligned}
\int_a^b f(x)dx &= (b-a)\int_0^1 f(\lambda(u))du \\
&\cong (b-a)[A_0 f(\lambda(0)) + A_1 f\left(\lambda\left(\frac{1}{3}\right)\right) + A_2 f\left(\lambda\left(\frac{2}{3}\right)\right) + A_3\lambda(1)] \\
&\approx \frac{1}{8}(b-a)[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)]
\end{aligned}
$$

If $h = (b-a)/3$, then

$$\int_a^b f(x)dx \approx \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)]$$

Error in Basic Simpson’s 1/3 Rule

Error in Simpson's 1/3 Rule

Theorem 2 (Error in Simpson's 1/3 Rule)
If $f \in C^4[a,b]$, then the error in Simpson's 1/3 rule is given by

$$-\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$$

for some $\xi \in (a,b)$

Proof:
Note that, when we approach the error through first interpolation error theorem we will get only an $O(\frac{(b-a)^4}{16})$ error term involving $f^{(3)}(\xi)$ as

$$f(x) - P_2(x) = \frac{1}{3!} f^{(3)}(\xi)(x-a)\left(x - \frac{a+b}{2}\right)(x-b)$$

Error in Simpson's 1/3 Rule

However, when we approach this using Taylor polynomial about x_1, then for each $x \in (x_0, x_2)$, there exist ξ such that

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{1}{2}f''(x_1)(x - x_1)^2 + \frac{1}{6}f^{(3)}(x_1)(x - x_1)^3 + \frac{1}{24}f^{(4)}(\xi)(x - x_1)^4$$

Integrating on both sides, we obtain that

$$\int_{x_0}^{x_2} f(x)dx = f(x_1)(x_2 - x_0) + \frac{1}{2}f'(x_1)[(x_2 - x_1)^2 - (x_0 - x_1)^2] + \frac{1}{6}f''(x_1)[(x_2 - x_1)^3 - (x_0 - x_1)^3] + \frac{1}{24}f^{(3)}(x_1)[(x_2 - x_1)^4 - (x_0 - x_1)^4] + \frac{1}{120}f^{(4)}(\xi)[(x_2 - x_1)^5 - (x_0 - x_1)^5]$$

Error in Simpson’s 1/3 Rule

Since

$$2(x_0 - x_1) = 2(x_1 - x_2) = (x_0 - x_2) = a - b,$$

we have

$$\int_{x_0}^{x_2} f(x)dx = f(x_1)(b-a) + \frac{1}{2}f'(x_1)\underbrace{[(x_2 - x_1)^2 - (x_0 - x_1)^2]}_{0}$$

$$+ \frac{1}{6}f''(x_1)\underbrace{[(x_2 - x_1)^3 - (x_0 - x_1)^3]}_{2(\frac{b-a}{2})^3} + \frac{1}{24}f^{(3)}(x_1)\underbrace{[(x_2 - x_1)^4 - (x_0 - x_1)^4]}_{0}$$

$$+ \frac{1}{120}f^{(4)}(\xi)\underbrace{[(x_2 - x_1)^5 - (x_0 - x_1)^5]}_{2(\frac{b-a}{2})^5}$$

Error in Simpson's 1/3 Rule

Since

$$f''(x) = 4\frac{f(x_0) - 2f(x_1) + f(x_2)}{(b-a)^2} - \frac{(b-a)^2}{12} f^{(4)}(\xi_1)$$

we have

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx &= f(x_1)(b-a) + \frac{1}{3}\left[\frac{f(x_0) - 2f(x_1) + f(x_2)}{(b-a)^2} - \frac{(b-a)^2}{12} f^{(4)}(\xi_1)\right]\left(\frac{b-a}{2}\right)^3 \\
&+ \frac{1}{60} f^{(4)}(\xi_2)\left(\frac{b-a}{2}\right)^5 \\
&= \left(\frac{b-a}{6}\right)[f(x_0) + 4f(x_1) + f(x_2)] - \left(\frac{b-a}{2}\right)^5\left[\frac{f^{(4)}(\xi_2)}{36} - \frac{f^{(4)}(\xi_1)}{60}\right] \\
&+ \frac{1}{60} f^{(4)}(\xi_2)\left(\frac{b-a}{2}\right)^5
\end{aligned}$$

Error in Simpson’s 1/3 Rule

$$\int_{x_0}^{x_2} f(x)dx = \left(\frac{b-a}{6}\right)[f(x_0) + 4f(x_1) + f(x_2)] - \frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$$

Hence the proof.
In terms of h, it written as

$$f(x) = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{1}{90}h^5 f^{(4)}(\xi)$$

Examples in Basic Simpson's 1/3 Rule

Error in Simpson's 1/3 Rule

Example 3

Evaluate

$$\int_0^1 e^{-x^2}dx$$

using Simpson's 1/3 rule.

$$a = 0, b = 1, \frac{a+b}{2} = 0.5,$$

$$f(a) = 1, f(b) = 0.36878, f\left(\frac{a+b}{2}\right) = 0.7788$$

$$\int_0^1 e^{-x^2}dx = \frac{1-0}{6}[f(0) + 4f(0.5) + f(1)] = 0.7472$$

Error in Simpson's 1/3 Rule

Example 4

Using Simpson's 1/3 rule, the following numerical integration can be obtained.

$$\int_0^2 x^2 dx = \tfrac{2-0}{6}[0+4+4] = 2.6667, \qquad E_t = \frac{8}{3} - \frac{8}{3} = 0$$

$$\int_0^2 x^4 dx = \tfrac{2-0}{6}[0+4+16] = 6.6667, \qquad E_t = \frac{32}{5} - \frac{20}{3} = \frac{-4}{15}$$

$$\int_0^2 \frac{1}{1+x} dx = \tfrac{2-0}{6}[1+2+\tfrac{1}{3}] = 1.1112, \qquad E_t = -0.0126$$

$$\int_0^2 \sqrt{1+x^2} dx = \tfrac{2-0}{6}[1+4*\sqrt{2}+\sqrt{5}] = 2.9643, \qquad E_t = -0.00641$$

$$\int_0^2 e^x dx = \tfrac{2-0}{6}[1+4*e+4*e^2] = 6.421, \qquad E_t = -0.032$$

Error in Basic Simpson's 1/3 Rule

Simpson's 3/8 Rule

Due to the fact that h is multiplied by $3/8$, it is referred as Simpson's 3/8 rule.

Theorem 5

Error in Simpson's 3/8 Rule

If $f \in C^4[a,b]$, then the error in Simpson's 3/8 rule is given by

$$-\frac{3}{80}\left(\frac{b-a}{3}\right)^5 f^{(4)}(\xi)$$

for some $\xi \in (a,b)$

Proof: Exercise

Some Closed Newton-Cotes Formula

Some Closed Newton-Cotes Formula

When $n = 1$, $(n+1)$-points closed Newton-Cotes formula provides trapezoidal rule. Similarly Simpson's 1/3 rule and 3/8 rules are obtained respectively while choosing $n = 2$ and $n = 3$. When $n = 4$, we obtain the following **Boole's rule**

$$\int_a^b f(x)dx \approx \frac{2h}{45}[7f(a) + 32f(a+h) + 12f(a+2h) + 32f(a+3h) + 7f(b)]$$

with error as

$$E_4(t) = -\frac{8}{945}h^7 f^{(6)}(\xi), \xi \in (a,b)$$

Some Closed Newton-Cotes Formula

When $n = 5$, we obtain the six-point Newton-Cotes closed rule as follows:

$$\int_a^b f(x)dx \approx \frac{5h}{28}[19f(a) + 75f(a+h) + 50f(a+2h) + 50f(a+3h) \\ + 75f(a+4h) + 19f(b)]$$

where the error is

$$E_5(t) = -\frac{275}{12096}h^7 f^{(6)}(\xi), \xi \in (a,b)$$

The following theorem gives the general error term for an $(n+1)$-points closed Newton-Cotes formula. From this theorem, you can notice that when n is even, we obtain higher accuracy compared to odd. Therefore, in practice, h is considered as $h = (b-a)/(2^k)$, where $n = 2^k$.

Closed Newton-Cotes Formula

Theorem 6

Suppose that $\sum_{i=0}^{n} A_i f(x_i)$ denotes the $(n+1)-$ points closed Newton-Cotes formula with $x_0 = a, x_n = b$ and $h = (b-a)/n$. Then $A_i = A_{n-i}$. If n is even, then the resulting formula is exact for any polynomials of degree $n+1$.

Error in Closed Newton-Cotes Formula

Theorem 7

Suppose that $\sum_{i=0}^{n} A_i f(x_i)$ denotes the $(n+1)-$ points closed Newton-Cotes formula with $x_0 = a, x_n = b$ and $h = (b-a)/n$. There exists $\xi \in (a,b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} A_i f(x_i) + E_n(t)$$

where the error is given by

$$E_n(t) = \begin{cases} \dfrac{h^{n+3}}{(n+2)!} f^{(n+2)}(\xi) \displaystyle\int_0^n t^2(t-1)\cdots(t-n)dt & n \text{ is even, } f \in C^{n+2}[a,b] \\ \dfrac{h^{n+2}}{(n+1)!} f^{(n+1)}(\xi) \displaystyle\int_0^n t(t-1)\cdots(t-n)dt & n \text{ is odd, } f \in C^{n+1}[a,b] \end{cases}$$

Open Newton-Cotes Formula

Open Newton-Cotes Formula

- The open Newton-Cotes formula do not include the endpoints $[a, b]$ as nodes.
- They use only interior nodes.
- The numerical approximation for open Newton-Cotes formula is same as (1), but, $x_i \in (a, b)$.
- In order to use the same formula (1), we redefine the points by spacing as follows: $x_{-1} = a, x_0 = a + h, x_{n+1} = b, x_i = x_0 + ih$ for each $i = 0, 1, 2, \cdots, n$ where $h = (b - a)/(n + 2)$.

Midpoint Rule

For the midpoint formula, we take $n = 0$, then $x_{-1} = a, x_0 = (a+b)/2, x_1 = b$. Therefore, the interior point is only $x_0 = (a+b)/2$. When we work on the interval $[0, 1]$, change of intervals and method of undetermined coefficients, we obtain that

$$\int_0^1 f(x)dx \cong A_0 f\left(\frac{1}{2}\right) \tag{10}$$

By the method of undetermined coefficients, we evaluate them on 1.

$$f(x) = 1 : \int_0^1 dx = A_0 \implies A_0 = 1$$

Midpoint Rule

The resulting formula is

$$\int_0^1 f(x)dx \approx f\left(\frac{1}{2}\right) \tag{11}$$

By (4), we obtain that

$$\begin{aligned}\int_a^b f(x)dx = (b-a)\int_0^1 f(\lambda(u))du &\cong (b-a)A_0 f\left(\lambda\left(\frac{1}{2}\right)\right)\\ &\approx (b-a)f\left(\frac{a+b}{2}\right)\end{aligned}$$

If $h = (b-a)/2$, then

$$\int_a^b f(x)dx \approx 2hf(a+h)$$

Midpoint Rule

Theorem 8 (Error in Midpoint Rule)
If $f \in C^2[a,b]$, then the error in Midpoint rule is given by

$$\frac{(b-a)^3}{24} f''(\xi)$$

Proof: By Taylor polynomial about $(a+b)/2$ for each $x \in (a,b)$ is given by

$$f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{1}{2} f''(\xi)\left(x - \frac{a+b}{2}\right)^2$$

Midpoint Rule

Integrating on both sides we obtain that

$$
\begin{aligned}
\int_a^b f(x)dx &= f\left(\frac{a+b}{2}\right)(b-a) + \frac{1}{2} f'\left(\frac{a+b}{2}\right) \underbrace{\left[\left(b - \frac{a+b}{2}\right)^2 - \left(a - \frac{a+b}{2}\right)^2\right]}_{0} \\
&\quad + \frac{1}{6} f''(\xi) \left[\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3\right] \\
&= f\left(\frac{a+b}{2}\right)(b-a) + \frac{1}{3} f''(\xi) \left(\frac{b-a}{2}\right)^3
\end{aligned}
$$

Hence the proof.

Two-Point Newton-Cotes Open Rule

For the Two-point formula, we take $n = 1$, then $x_{-1} = a, x_0 = (a+b)/3, x_0 = 2(a+b)/3, x_2 = b$. Therefore, the interior points are only $x_0 = (a+b)/3$ and $x_1 = 2(a+b)/3$. When we work on the interval $[0, 1]$, change of intervals and method of undetermined coefficients, we obtain that

$$\int_0^1 f(x)dx \cong A_0 f\left(\frac{1}{3}\right) + A_1 f\left(\frac{2}{3}\right) \tag{12}$$

Two-Point Newton-Cotes Open Rule

By the method of undetermined coefficients, we evaluate them on $\{1, x\}$.

$$f(x) = 1 : \int_0^1 dx = A_0 + A_1 \implies A_0 + A_1 = 1$$

$$f(x) = x : \int_0^1 x dx = A_0 \frac{1}{3} + A_1 \frac{2}{3} \implies \frac{1}{3} A_0 + \frac{2}{3} A_1 = \frac{1}{2}$$

Solving for A_0, A_1, we obtain that $A_0 = 1/2, A_1 = 1/2$.

Two-Point Newton-Cotes Open Rule

The resulting formula is

$$\int_0^1 f(x)dx \approx \frac{1}{2}f\left(\frac{1}{3}\right) + \frac{1}{2}f\left(\frac{2}{3}\right) \tag{13}$$

By (4), we obtain that

$$\begin{aligned}\int_a^b f(x)dx &= (b-a)\int_0^1 f(\lambda(u))du \\ &\cong (b-a)A_0 f\left(\lambda\left(\frac{1}{3}\right)\right) + (b-a)A_1 f\left(\lambda\left(\frac{2}{3}\right)\right) \\ &\approx \frac{(b-a)}{2}\left[f\left(a+\frac{b-a}{3}\right) + f\left(a+2\frac{b-a}{3}\right)\right]\end{aligned}$$

Two-Point Newton-Cotes Open Rule

If $h = (b-a)/3$, then

$$\int_a^b f(x)dx \approx \frac{3h}{2}[f(a+h) + f(a+2h)]$$

Two-Point Newton-Cotes Open Rule

Theorem 9 (Error in Two-point Newton-Cotes Open Rule)
If $f \in C^2[a, b]$, then the error in Two-point rule is given by

$$\frac{(b-a)^3}{36} f''(\xi)$$

for some $\xi \in (a, b)$

Midpoint Rule

Example 10

Evaluate

$$\int_0^1 e^{-x^2}\,dx$$

using Midpoint rule.

$$a = 0, b = 1, \frac{a+b}{2} = 0.5, f\left(\frac{a+b}{2}\right) = 0.7788$$

$$\int_0^1 e^{-x^2}\,dx = (1-0)f(0.5) = 0.7788$$

Two-Point Newton-Cotes Open Rule

Example 11

Using Two-Point Newton-Cotes open rule, the following numerical integration can be obtained.

$$
\begin{aligned}
\int_0^2 x^2 dx &= \tfrac{3-0}{2}\left[\tfrac{4}{9}+\tfrac{16}{9}\right] = \tfrac{10}{3}, && E_t = \frac{2}{3} \\
\int_0^3 \frac{1}{1+x} dx &= \tfrac{3-0}{2}\left[0.5+\tfrac{1}{3}\right] = 1.25, && E_t = 0.1363 \\
\int_0^3 \sqrt{1+x^2} dx &= \tfrac{3-0}{2}\left[\sqrt{2}+\sqrt{5}\right] = 5.0673, && E_t = -0.5853 \\
\int_0^3 e^x dx &= \tfrac{3-0}{2}\left[e+e^2\right] = 15.16, && E_t = 3.9246
\end{aligned}
$$

Some Open Newton-Cotes Formula

Some Open Newton-Cotes Formula

When $n = 0$, open Newton-Cotes formula provides Midpoint rule. Similarly, when $n = 1$, it yields the two-point Newton-Cotes open rule. When $n = 3$, the following three point Newton-Cotes open rule is obtained

$$\int_a^b f(x)dx \approx \frac{4h}{3}[2f(a+h) - f(a+2h) + 2f(a+3h)]$$

with error as

$$E_3(t) = \frac{14}{45}h^5 f^{(4)}(\xi), \xi \in (a, b)$$

Some Open Newton-Cotes Formula

When $n = 4$, we obtain the following four-point Newton-Cotes open rule

$$\int_a^b f(x)dx \approx \frac{5h}{24}[11f(a+h) + f(a+2h) + f(a+3h) + 11f(a+4h)]$$

where the error is

$$E_4(t) = \frac{95}{144}h^5 f^{(4)}(\xi), \xi \in (a,b)$$

Some Open Newton-Cotes Formula

When $n = 5$, we obtain the following five-point Newton-Cotes open rule

$$\int_a^b f(x)dx \approx \frac{6h}{20}[11f(a+h) - 14f(a+2h) + 26f(a+3h) - 14f(a+4h) + 11f(a+5h)]$$

where the error is

$$E_5(t) = -\frac{41}{140}h^7 f^{(6)}(\xi), \xi \in (a,b)$$

The following theorem gives the general error term for an $(n+1)$-points closed Newton-Cotes formula.

Some Open Newton-Cotes Formula

Theorem 12

Suppose that $\sum_{i=0}^{n} A_i f(x_i)$ denotes the open Newton-Cotes formula with $x_{-1} = a, x_{n+1} = b, x_0 = a+h, x_i = x_0 + ih, i = 0, 1, 2, \cdots, n$ where $h = (b-a)/(n+2)$. There exists $\xi \in (a, b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} A_i f(x_i) + E_n(t)$$

where the error is given by

$$E_n(t) = \begin{cases} \frac{h^{n+3}}{(n+2)!} f^{(n+2)}(\xi) \int_{-1}^{n+1} t^2(t-1)\cdots(t-n)dt & n \text{ is even, } f \in C^{n+2} \\ \frac{h^{n+2}}{(n+1)!} f^{(n+1)}(\xi) \int_{-1}^{n+1} t(t-1)\cdots(t-n)dt & n \text{ is odd, } f \in C^{n+1} \end{cases}$$

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in

MA633L-Numerical Analysis

Lecture 32 : Numerical Integration - Newton-Cotes

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

April 4, 2025