MA633L-Numerical Analysis

Lecture 39 : Numerical Differentiation - Higher Order Taylor's Method

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Higher Order Taylor Series Method

IVP

The first order differential equation

$$\frac{dy}{dt} = f(t, y(t)), t_0 \leq t \leq T \tag{1}$$

with an initial condition

$$y(t_0) = y_0 \tag{2}$$

Taylor Series

Basic building block of numerical method: Taylor series.
Taylor series of y at $t+h$, then we have

$$y(t+h) = y(t) + hy'(t) + \frac{1}{2!}h^2y''(t) + \frac{1}{3!}h^3y'''(t) + \cdots + \frac{1}{m!}h^m y^{(m)}(t) + \cdots \quad (3)$$

When $m = 1$, we obtain the Taylor series method of order 1, that is called Euler's method.

$$\int_t^{t+h} f(t, y(t))dt \approx hf(t, y(t)),$$

Higher Order Taylor Series Method

Example 1

Apply Taylor's method of order (a) two and (b) four with $n = 10$ to the IVP $y' = 1 - t^2 + y \quad 0 \leq t \leq 2, y(0) = 0.5.$

Higher Order Taylor Series Method

Solution

$$y(t+h) = y(t) + hy'(t) + \frac{1}{2!}h^2y''(t) + \frac{1}{3!}h^3y'''(t) + \cdots + \frac{1}{m!}h^my^{(m)}(t) + \cdots \quad (4)$$

For $m = 2$, we have

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2!}[f_t + f_y f]$$

Now, $f_t = -2t$, $f_y = 1$

$$y(t_{i+1}) = y(t_i) + h(1 - t_i^2 + y(t_i)) + \frac{h^2}{2!}[1 - t_i^2 + y(t_i) - 2t_i]$$

$$y(t_{i+1}) = y(t_i) + \left(h + \frac{h^2}{2}\right)[1 - t_i^2 + y(t_i) - 2t_i] - 2h^2t_i$$

Higher Order Taylor Series Method

t_i	$y(t_i)$	$y_{exact}(t_i)$	E_t
0.0	0.5	0.5	0.0000
0.2	0.83	0.8292	0.0292
0.4	1.2158	1.214	0.0017
0.6	1.6521	1.6489	0.0031
0.8	2.1323	2.1272	0.0051
1.0	2.6486	2.6408	0.0077
1.2	3.1913	3.1799	0.0114
1.4	3.7486	3.7324	0.0162
1.6	4.3061	4.2834	0.0226
1.8	4.8462	4.8151	0.0311
2.0	5.3476	5.3054	0.0422

Higher Order Taylor Series Method

For $m = 4$, we have

$$
\begin{aligned}
f' &= y - t^2 + 1 - 2t \\
f'' &= y' - 2t - 2 = y - t^2 - 2t - 1 \\
f''' &= y' - 2t - 2 = y - t^2 - 2t - 1
\end{aligned}
$$

Higher Order Taylor Series Method

$$
\begin{aligned}
y(t_{i+1}) &= y(t_i) + h(1 - t_i^2 + y(t_i)) + \frac{h^2}{2!}[1 - t_i^2 + y(t_i) - 2t_i] \\
&+ \frac{h^3}{3!}[y - t_i^2 - 2t_i - 1] + \frac{h^4}{4!}[y - t_i^2 - 2t_i - 1] \\
y(t_{i+1}) &= y(t_i) + \left(h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right)[y(t_i) - t_i^2] \\
&- \left(1 + \frac{h}{3} + \frac{h^2}{12}\right) h^2 t_i + h + \frac{h^2}{2} - \frac{h^3}{6} - \frac{h^4}{24}
\end{aligned}
$$

Higher Order Taylor Series Method

t_i	$y(t_i)$	$y_{exact}(t_i)$	E_t
0.0	0.5	0.5	0.0000
0.2	0.8293	0.8292	0.000001
0.4	1.2141	1.214	0.000003
0.6	1.6489	1.6489	0.000006
0.8	2.1272	2.1272	0.000010
1.0	2.6408	2.6408	0.000015
1.2	3.1799	3.1799	0.000023
1.4	3.7324	3.7324	0.000032
1.6	4.2832	4.2834	0.000045
1.8	4.8152	4.8151	0.000062
2.0	5.3055	5.3054	0.000083

Higher Order Taylor Series Method

Theorem 2

If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad t_0 \le t \le T, y(t_0) = y_0$$

with step size h and if $y \in C^{n+1}[t_0, T]$, the the local truncation error is $O(h^n)$.

Runge-Kutta Method

Runge-Kutta Method

Taylor series. But, Taylor series in two variables. The infinite series is

$$f(t+h, y+k) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} \right)^i f(t, y) \tag{5}$$

Runge-Kutta Method

$$\left(h\frac{\partial}{\partial t}+k\frac{\partial}{\partial y}\right)^0 f(t,y)=f$$

$$\left(h\frac{\partial}{\partial t}+k\frac{\partial}{\partial y}\right)^1 f(t,y)=h\frac{\partial f}{\partial t}+k\frac{\partial f}{\partial y}$$

$$\left(h\frac{\partial}{\partial t}+k\frac{\partial}{\partial y}\right)^2 f(t,y)=h^2\frac{\partial^2 f}{\partial t^2}+2hk\frac{\partial^2 f}{\partial t\partial y}+k^2\frac{\partial^2 f}{\partial y^2}$$

Runge-Kutta Method

The truncated Taylor series in two variables is given by

$$f(t+h, y+k) = \sum_{i=0}^{n-1} \frac{1}{i!} \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} \right)^i f(t, y)$$
$$+ \frac{1}{n!} \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} \right)^n f(\xi, \zeta)$$

where (ξ, ζ) lies on the line segment that joins (t, y) to $(t+h, y+k)$.

Runge-Kutta Method or Order 2

In the Runge-Kutta method of order 2, a formula is adopted that has two function evaluations of the special form

$$K_1 = hf(t, y)$$
$$K_2 = hf(t + \alpha h, y + \beta K_1)$$

and a linear combination of K_1 and K_2 is added to the value of y at t to obtain the values at $t + h$

$$y(t + h) = y(t) + w_1 K_1 + w_2 K_2$$

Runge-Kutta Method or Order 2

$$y(t+h) = y(t) + w_1 h f(t,y) + w_2 h f(t + \alpha h, y + \beta h f(t,y))$$

$$f(t+\alpha h, y + \beta h f) = f + \alpha h f_t + \beta h f f_y + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h f \frac{\partial}{\partial y}\right)^2 f(\xi, \zeta)$$

Truncated Taylor series

$$y(t+h) = y(t) + (w_1 + w_2) h f(t,y) + \alpha w_2 h^2 f_t + \beta w_2 h^2 f f_y + O(h^3)$$

$$y(t+h) = y(t) + h y'(t) + \frac{1}{2!} h^2 y''(t) + \frac{1}{3!} h^3 y'''(t) + \cdots + \frac{1}{m!} h^m y^{(m)}(t) + \cdots \quad (6)$$

Runge-Kutta Method or Order 2

By Comparing

$$w_1 + w_2 = 1$$
$$\alpha w_2 = \frac{1}{2}$$
$$\beta w_2 = \frac{1}{2}$$

Four unknowns, but three equations. Conveniently, let us choose $\alpha = 1$, then

$$\implies w_1 = \frac{1}{2}, w_2 = \frac{1}{2} \implies \beta = 1$$

The resulting second-order Runge-Kutta method is given by

$$y(t+h) = y(t) + \frac{h}{2} f(t, y) + \frac{h}{2} f(t+h, y+hf(t, y)) \tag{7}$$

Runge-Kutta Method or Order 2

$$\boxed{y(t+h) = y(t) + \frac{1}{2}(K_1 + K_2)}$$

where

$$K_1 = hf(t, y)$$
$$K_2 = hf(t+h, y+K_1)$$

Since the selection of α is arbitrary, in particular, if we choose $\alpha = \beta$, we have

$$w_1 = 1 - \frac{1}{2\alpha}, w_2 = \frac{1}{2\alpha}$$

Runge-Kutta Method or Order 2

Theorem 3

The error term for Runge-Kutta methods of order 2 is

$$\frac{h^3}{4}\left(\frac{2}{3}-\alpha\right)\left(\frac{\partial}{\partial t}+f\frac{\partial}{\partial y}\right)^2 f+\frac{h^3}{6}f_y\left(\frac{\partial}{\partial t}+f\frac{\partial}{\partial y}\right)f$$

That is $O(h^3)$.

Proof:

A general explicit Runge-Kutta method of order 2 is given by:

$$\begin{aligned}
k_1 &= f(t,y),\\
k_2 &= f(t+\alpha h, y+\alpha h k_1),\\
y_{n+1} &= y + h\left(\left(1-\frac{1}{2\alpha}\right)k_1+\frac{1}{2\alpha}k_2\right),
\end{aligned}$$

where α is a method parameter.

Runge-Kutta Method or Order 2

We expand the exact solution at $t + h$:

$$y(t+h) = y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y^{(3)} + \mathcal{O}(h^4)$$

Using the chain rule:

$$\begin{aligned} y' &= f, \\ y'' &= f_t + f_y f, \\ y^{(3)} &= \left(\frac{d}{dt}\right)^2 f = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)^2 f. \end{aligned}$$

Runge-Kutta Method or Order 2

We expand K_2:

$$
\begin{aligned}
K_2 &= f(t + \alpha h, y + \alpha h f) \\
&= f + \alpha h \left(f_t + f f_y\right) + \frac{(\alpha h)^2}{2} \left(f_{tt} + 2 f f_{ty} + f^2 f_{yy}\right) + \mathcal{O}(h^3)
\end{aligned}
$$

Substitute into the RK2 formula:

$$
\begin{aligned}
y_{n+1} &= y + h \left(\left(1 - \frac{1}{2\alpha}\right) f + \frac{1}{2\alpha} K_2 \right) \\
&= y + hf + \frac{h^2}{2}(f_t + f f_y) + \frac{\alpha h^3}{4} \left(f_{tt} + 2 f f_{ty} + f^2 f_{yy}\right) + \mathcal{O}(h^4)
\end{aligned}
$$

Runge-Kutta Method or Order 2

Compare the true solution with the RK2 approximation:

$$
\begin{aligned}
\text{LTE} &= y(t+h) - y_{n+1} \\
&= \left(y + hf + \frac{h^2}{2}(f_t + f f_y) + \frac{h^3}{6}\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)^2 f \right) \\
&\quad - \left(y + hf + \frac{h^2}{2}(f_t + f f_y) + \frac{\alpha h^3}{4}\left(f_{tt} + 2f f_{ty} + f^2 f_{yy}\right) \right) + \mathcal{O}(h^4)
\end{aligned}
$$

Simplify the cubic term difference:

$$
\begin{aligned}
\text{LTE} &= \left(\frac{1}{6} - \frac{\alpha}{4}\right) h^3 \left(f_{tt} + 2f f_{ty} + f^2 f_{yy}\right) + \mathcal{O}(h^4) \\
&= \frac{h^3}{4}\left(\frac{2}{3} - \alpha\right)\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)^2 f + \frac{h^3}{6} f_y \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right) f + \mathcal{O}(h^4)
\end{aligned}
$$

Hence the Proof.

Runge-Kutta Method or Order 3

In the Runge-Kutta method of order 3 we need to evaluate the function three times. Its formula is given by

$$\boxed{y(t+h) = y(t) + \frac{1}{9}(2K_1 + 3K_2 + 4K_3)} \tag{8}$$

where

$$K_1 = hf(t, y)$$
$$K_2 = hf(t + \frac{1}{2}h, y + \frac{1}{2}K_1)$$
$$K_3 = hf(t + \frac{3}{4}h, y + \frac{3}{4}K_2)$$

Runge-Kutta Method or Order 4

Runge-Kutta method of order 2 is not widely used in large computers as its error is only $O(h^3)$. The classical fourth order Runge-Kutta method is widely used in IVP. Its formula is given by

$$y(t+h) = y(t) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{9}$$

where

$$\begin{aligned} K_1 &= hf(t, y) \\ K_2 &= hf(t + \frac{1}{2}h, y + \frac{1}{2}K_1) \\ K_3 &= hf(t + \frac{1}{2}h, y + \frac{1}{2}K_2) \\ K_4 &= hf(t + h, y + K_3) \end{aligned}$$

The derivation of fourth order is very tedious. If you would like to know, refer Henrici book.

Runge-Kutta Method

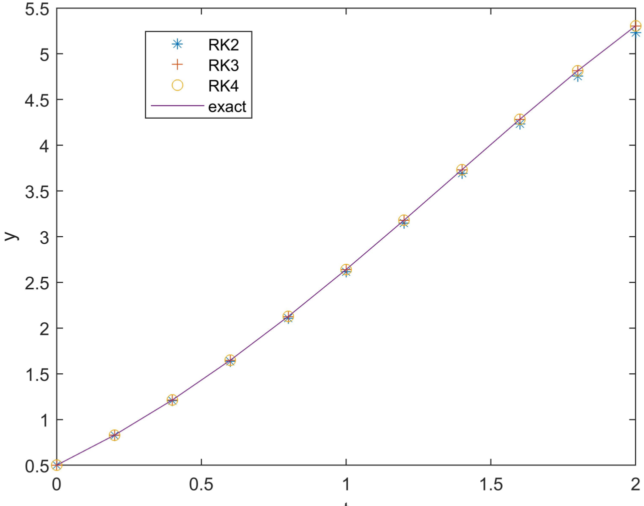
Example 4
Using the RK2,RK3 and RK4 method, solve the following IVP: $y' = 1 - t^2 + y \quad 0 \leq t \leq 2, y(0) = 0.5$. with $n = 10$

Runge-Kutta Method

The following table shows RK2, RK3 and RK4 methods approximations and errors

t_i	$y_{exact}(t_i)$	$RK2(t_i)$	E_t	$RK3(t_i)$	E_t	$RK4(t_i)$	E_t
0	0.5000	0.5000	0	0.5000	0	0.5000	0
0.2000	0.8293	0.8260	0.0033	0.8292	0.0001	0.8293	0.0000
0.4000	1.2141	1.2069	0.0072	1.2139	0.0002	1.2141	0.0000
0.6000	1.6489	1.6372	0.0117	1.6486	0.0003	1.6489	0.0000
0.8000	2.1272	2.1102	0.0170	2.1267	0.0005	2.1272	0.0000
1.0000	2.6409	2.6177	0.0232	2.6402	0.0006	2.6408	0.0000
1.2000	3.1799	3.1496	0.0304	3.1791	0.0008	3.1799	0.0000
1.4000	3.7324	3.6937	0.0387	3.7314	0.0010	3.7323	0.0001
1.6000	4.2835	4.2351	0.0484	4.2822	0.0013	4.2834	0.0001
1.8000	4.8152	4.7556	0.0596	4.8137	0.0015	4.8151	0.0001
2.0000	5.3055	5.2331	0.0724	5.3037	0.0017	5.3054	0.0001

Runge-Kutta Method



Multistep Methods

Multistep Method

Definition 5 (m-step multistep method)

An **m-step multistep method** for solving the IVP (**??**) has a difference function for finding the approximation $y(t_{i+1})$ represented by the following equation, where m is an integer greater than 1:

$$\begin{aligned} y(t_{i+1}) &= \sum_{j=0}^{m-1} a_j y(t_{i-(m-j-1)}) + h\left[\sum_{j=0}^{m-1} b_j f(t_{i-(m-j-1)}, y(t_{i-(m-j-1)}))\right] \\ &\quad + h b_m f(t_{i+1}, y(t_{i+1})) \end{aligned}$$

for $i = m-1, m, \cdots, n-1$, where $h = (T - t_0)/n$, $a_j's$ and $b_j's$ are constants and $y_0, y(t_1), y(t_2) \cdots, y(t_{m-1})$ are specified.

Multistep Method

$$
\begin{aligned}
y(t_{i+1}) = {} & a_{m-1}y_i + a_{m-2}y_{i-1} + \cdots + a_0 y_{i+1-m} \\
& + h[b_{m-1}f(t_i, y_i) + b_{m-2}f(t_{i-1}, y_{i-1}) \\
& + \cdots + b_0 f(t_{i+1-m}, y_{i+1-m})] \\
& + h b_m f(t_{i+1}, y(t_{i+1}))
\end{aligned}
$$

When $b_m = 0$, the method is called explicit or open as $y(t_{i+1})$ is written explicitly in terms of previous values. When $b_m \neq 0$, it is called implicit method or closed as $y(t_{i+1})$ occur on both sides of the equation.

Multistep Method

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)dt = \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

$$\implies y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

Since f is continuous, we can use a corresponding interpolating polynomial $P(t)$ to $f(t, y(t))$ using data points

$(t_i, f(t, y(t_i))), (t_{i-1}, f(t_{i-1}, y(t_{i-1}))), \cdots, (t_{i-(m-1)}, f(t_{i-(m-1)}, y(t_{i-(m-1)})))$.

Since $P(t)$ is the interpolating polynomial of $f(t, y(t))$, we have

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} P(t)dt$$

Newton backward or forward difference for interpolating polynomial.

Adams-Bashforth explicit m-step methods - backward difference polynomial $P_{m-1}(t)$.

Multistep Method

Multistep method can be written as

$$
\begin{aligned}
y(t_{i+1}) = y(t_i) + h \Bigg[& f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i)) \\
& + \frac{3}{8}\nabla^3 f(t_i, y(t_i)) + \frac{251}{720}\nabla^4 f(t_i, y(t_i)) \\
& + \frac{95}{288}\nabla^5 f(t_i, y(t_i)) \Bigg] + K h^{m+1} f^{(m)}(\xi_i, y(\xi_i))
\end{aligned} \tag{10}
$$

Adams-Bashforth method

Adams-Bashforth method

It is an explicit method. Therefore, $y_0, y_1, \cdots y_{m-1}$ must be specified. Further, $b_{m-1}, \cdots, b_0$ must be specified. $a_{m-1} = 1$ and $a_0 = a_1 = \cdots = a_{m-2} = 0 = b_m$.

Adams-Bashforth Two-Step Explicit Method

Specify y_0 and y_1, then compute . Let us derive two-step Adams-Bashforth method from equation (10). Here $m = 2$. Therefore we have

$$
\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]\right] \\
&= y(t_i) + \frac{h}{2}\left[3f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))\right]
\end{aligned}
$$

$$
\boxed{y(t_{i+1}) = y(t_i) + \frac{h}{2}[3f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]} \tag{11}
$$

where $i = 1, 2, \cdots, n-1$. The error is $\frac{5}{12}y'''(\xi_i)h^2$ for some $\xi_i \in (t_{i-1}, t_{i+1})$.

Adams-Bashforth Three-Step Explicit Method

Specify y_0, y_1 and y_2, then compute. Let us derive three-step Adams-Bashforth method from equation (10) Here $m = 3$. Therefore we have

$$
\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]\right. \\
&\left. + \frac{5}{12}[f(t_i, y(t_i)) - 2f(t_{i-1}, y(t_{i-1})) + f(t_{i-2}, y(t_{i-2}))]\right] \\
&= y(t_i) + \frac{h}{12}\left[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1}))\right. \\
&\left. + 5f(t_{i-2}, y(t_{i-2}))\right]
\end{aligned}
$$

Adams-Bashforth Three-Step Explicit Method

$$y(t_{i+1}) = y(t_i) + \frac{h}{12}[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) \tag{12}$$

$$+ 5f(t_{i-2}, y(t_{i-2}))] \tag{13}$$

where $i = 2, 3 \cdots, n-1$. The error is $\frac{3}{8}y^{(4)}(\xi_i)h^3$ for some $\xi_i \in (t_{i-2}, t_{i+1})$.

Adams-Bashforth Four-Step Explicit Method

Specify y_0, y_1, y_2 and y_3, then we can derive the following from (10)

$$y(t_{i+1}) = y(t_i) + \frac{h}{24}[55f(t_i, y(t_i)) - 59f(t_{i-1}, y(t_{i-1})) \tag{14}$$

$$+ 37f(t_{i-2}, y(t_{i-2})) - 9f(t_{i-3}, y(t_{i-3}))] \tag{15}$$

where $i = 3, 4, \cdots, n-1$. The error is $\frac{251}{720}y^{(5)}(\xi_i)h^4$ for some $\xi_i \in (t_{i-3}, t_{i+1})$.

Adams-Bashforth Five-Step Explicit Method

Specify y_0, y_1, y_2, y_3 and y_4, then we can derive the following equation from (10).

$$
\begin{aligned}
y(t_{i+1}) = y(t_i) + \frac{h}{720}[&1901 f(t_i, y(t_i)) - 2774 f(t_{i-1}, y(t_{i-1})) \\
&+ 2616 f(t_{i-2}, y(t_{i-2})) - 1274 f(t_{i-3}, y(t_{i-3})) \\
&+ 251 f(t_{i-4}, y(t_{i-4}))]
\end{aligned}
$$

where $i = 4, 5, \cdots, n-1$. The error is $\frac{95}{288} y^{(6)}(\xi_i) h^4$ for some $\xi_i \in (t_{i-4}, t_{i+1})$.

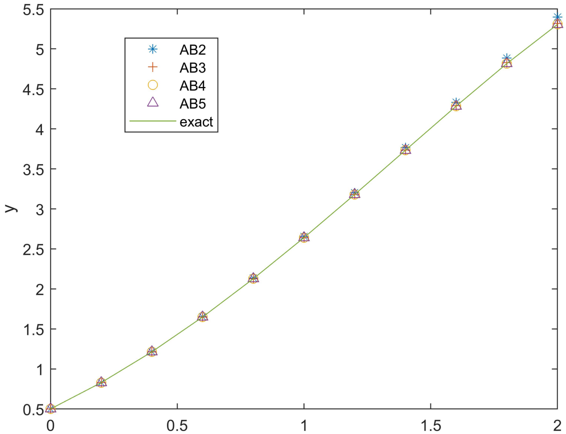
Adams-Bashforth

Example 6

Using the Adams-Bashforth two-,three-,four- and five-step explicit methods, solving the following IVP: $y' = 1 - t^2 + y \quad 0 \leq t \leq 2, y(0) = 0.5$. use the following information appropriately $y(0.2) = 0.8293, y(0.4) = 1.2141, y(0.6) = 1.6489, y(0.8) = 2.1272$ with $n = 10$.

Adams-Bashforth

The following table shows Adams-Bashforth two-,three-,four- and five-steps explicit methods approximations and errors



Adams-Bashforth

t_i	$y_{exact}(t_i)$	$AB2(t_i)$	E_t	$AB3(t_i)$	E_t	$AB4(t_i)$	E_t	$AB5(t_i)$	E_t
0	0.5000	0.5000	0	0.5000	0	0.5000	0	0.5000	0
0.2000	0.8293	0.8293	0.0000	0.8293	0.0000	0.8293	0.0000	0.8293	0.0000
0.4000	1.2141	1.2161	0.0020	1.2141	0.0000	1.2141	0.0000	1.2141	0.0000
0.6000	1.6489	1.6540	0.0050	1.6494	0.0004	1.6489	0.0000	1.6489	0.0000
0.8000	2.1272	2.1366	0.0093	2.1283	0.0011	2.1272	0.0000	2.1272	0.0000
1.0000	2.6409	2.6561	0.0153	2.6428	0.0020	2.6410	0.0002	2.6409	0.0000
1.2000	3.1799	3.2033	0.0234	3.1831	0.0032	3.1803	0.0003	3.1800	0.0000
1.4000	3.7324	3.7667	0.0343	3.7373	0.0049	3.7330	0.0006	3.7325	0.0001
1.6000	4.2835	4.3324	0.0489	4.2906	0.0071	4.2844	0.0009	4.2836	0.0001
1.8000	4.8152	4.8835	0.0683	4.8254	0.0102	4.8165	0.0013	4.8154	0.0002
2.0000	5.3055	5.3992	0.0938	5.3197	0.0142	5.3074	0.0019	5.3057	0.0003

Adams-Moulton method

Adams-Moulton technique

Adams-Moulton method is obtained by using an additional interpolation node $(t_{i+1}, f(t_{i+1}, y(t_{i+1})))$ in the approximation of the integral

$$\int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

It is an implicit method. We have to specify $y_0, y_1, y_2, \cdots, y_{m-1}$. Further $b_m, b_{m-1}, \cdots, b_0$ must be specified. $a_{m-1} = 1$ and $a_0 = a_1 = \cdots = a_{m-2} = 0$.

Adams-Moulton One-Step Implicit Method

Here y_0 must be specified. By employing trapezoidal rule for the integration produces the following result. Here $m = 1$. Therefore we have

$$\begin{aligned} y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i))\right] \\ &= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_{i+1}, y(t_{i+1})) - f(t_i, y(t_i))]\right] \\ &= y(t_i) + \frac{h}{2}[f(t_{i+1}, y(t_{i+1})) + f(t_i, y(t_i))] \end{aligned}$$

$$\boxed{y(t_{i+1}) = y(t_i) + \frac{h}{2}[f(t_{i+1}, y(t_{i+1})) + f(t_i, y(t_i))]} \quad (16)$$

where $i = 0, 1, 2, \cdots, n-1$. The error is $-\frac{1}{24}y^{(4)}(\xi_i)h^3$ for some $\xi_i \in (t_{i-1}, t_{i+1})$.

Adams-Moulton Two-Step Implicit Method

Here y_0 and y_1 must be specified. Here $m = 2$. Therefore we have

$$\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]\right. \\
&\quad \left. + \frac{5}{12}[f(t_{i+1}, y(t_{i+1})) - 2f(t_i, y(t_i)) + f(t_{i-1}, y(t_{i-1}))]\right] \\
&= y(t_i) + \frac{h}{12}\left[5f(t_{i+1}, y(t_{i+1})) + 8f(t_i, y(t_i)) - 5f(t_{i-1}, y(t_{i-1}))\right]
\end{aligned}$$

$$\boxed{y(t_{i+1}) = y(t_i) + \frac{h}{12}[5f(t_{i+1}, y(t_{i+1})) + 8f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]} \qquad (17)$$

where $i = 1, 2, \cdots, n-1$. The error is $-\frac{1}{24}y^{(4)}(\xi_i)h^3$ for some $\xi_i \in (t_{i-1}, t_{i+1})$.

Adams-Moulton Three-Step Implicit Method

We must specify y_0, y_1 and y_2 and the formula is

$$y(t_{i+1}) = y(t_i) + \frac{h}{24}[9f(t_{i+1}, y(t_{i+1})) + 19f(t_i, y(t_i)) \\ - 5f(t_{i-1}, y(t_{i-1})) + f(t_{i-2}, y(t_{i-2}))]$$

where $i = 2, 3, \cdots, n-1$. The error is $-\frac{19}{720}y^{(5)}(\xi_i)h^4$ for some $\xi_i \in (t_{i-2}, t_{i+1})$.

Adams-Moulton Four-Step Implicit Method

We must specify y_0, y_1, y_2 and y_3 and the formula is

$$y(t_{i+1}) = y(t_i) + \frac{h}{720}[251f(t_{i+1}, y(t_{i+1})) + 646f(t_i, y(t_i)) \\ - 264f(t_{i-1}, y(t_{i-1})) + 106f(t_{i-2}, y(t_{i-2})) - 19f(t_{i-3}, y(t_{i-3}))]$$

where $i = 3, 4, \cdots, n-1$. The error is $-\frac{3}{160}y^{(6)}(\xi_i)h^5$ for some $\xi_i \in (t_{i-3}, t_{i+1})$.

Thanks

Doubts and Suggestions

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भारतीय प्रौद्योगिकी संस्थान तिरुपति

TIRUPATI

MA633L-Numerical Analysis

Lecture 39 : Numerical Differentiation - Higher Order Taylor's Method

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