MA633L-Numerical Analysis

Lecture 4 : Master Theorem and Order of Convergence

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Master Theorem: Proof

Master Theorem

Theorem 1

Consider the recurrence relation

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is asymptotically positive function. Then

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = O(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if f satisfies the smoothness condition $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$, then $T(n) = \Theta(f(n))$.

Application: Matrix-Matrix Multiplication

How many operations are required for matrix-matrix multiplication where A and B are matrices of size $n \times n$. For any two entries from A and B, we require $O(1)$ time. The total runtime or computation time is $O(n^3)$

Divide and Conquer: If A is divided into 4 submatrices and B is divided into 4 submatrices as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Application: Matrix-Matrix Multiplication

What is the recurrence relation for this approach?

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$T(n) = 8T\left(\frac{n}{2}\right) + O(n^2)$$

Comparing it with Master theorem, $a = 8, b = 2$ and $f(n) = O(n^2)$.
Therefore,

$$T(n) = O(n^{\log_b a}) = O(n^{\log_2 8}) = O(n^3)$$

No improvement! Let us see the proof of Master Theorem before we can improve it!

Master Theorem: Proof

The following observation is easy to prove and left as an exercise

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) + O(n^{\log_b a}) \tag{1}$$

Proof of (1):

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) + O(n^{\log_b a}) \leq \sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \varepsilon} + O(n^{\log_b a}) \tag{2}$$

If we claim the following, then proof of (1) follows immediately

$$\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \varepsilon} = O(n^{\log_b a})$$

Master Theorem: Proof

Claim:

$$\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon} = O(n^{\log_b a})$$

$$\begin{aligned}
\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\varepsilon} &= n^{\log_b a-\varepsilon}\sum_{i=0}^{\log_b n} a^i b^{-i\log_b a} b^{i\varepsilon} = n^{\log_b a-\varepsilon}\sum_{i=0}^{\log_b n} a^i a^{-i} b^{i\varepsilon} \\
&= n^{\log_b a-\varepsilon}\sum_{i=0}^{\log_b n} b^{i\varepsilon} = n^{\log_b a-\varepsilon}\frac{b^{\varepsilon(\log_b n+1)}-1}{b^\varepsilon-1} \\
&= n^{\log_b a-\varepsilon}\frac{n^\varepsilon b^\varepsilon-1}{b^\varepsilon-1} \leq n^{\log_b a-\varepsilon}\frac{n^\varepsilon b^\varepsilon}{b^\varepsilon-1} = n^{\log_b a}\frac{b^\varepsilon}{b^\varepsilon-1} \\
&= O(n^{\log_b a})
\end{aligned}$$

Master Theorem: Proof

Proof of (2): In the above proof, use $\epsilon = 0$

$$\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a} = n^{\log_b a} \sum_{i=0}^{\log_b n} b^{i\varepsilon} = n^{\log_b a} \sum_{i=0}^{\log_b n} 1$$

$$= n^{\log_b a}(\log_b n + 1) = \Theta(n^{\log_b a} \log_b n)$$

Hence

$$T(n) = \Theta(n^{\log_b a} \log_b n) + O(n^{\log_b a}) = \Theta(n^{\log_b a} \log_b n)$$

Master Theorem: Proof

Proof of (3): We need to prove that

$$C_1 f(n) \leq T(n) \leq C_2 f(n)$$

Lower bound is easy to prove and left as an exercise. For the upper bound, we need the additional condition, that is smoothness condition. The smoothness condition is satisfied by

$$f(n) = n^{\log_b a+\varepsilon}$$

for any $\epsilon > 0$ with $c = b^{-\epsilon} < 1$.

Master Theorem: Proof

Proof of (3):

$$af\left(\frac{n}{b}\right) = a\left(\frac{n}{b}\right)^{\log_b a+\varepsilon} = an^{\log_b a+\varepsilon}b^{-\log_b a}b^{-\varepsilon} = f(n)b^{-\varepsilon}$$

In this case,

$$a^i f\left(\frac{n}{b^i}\right) \le c^i f(n)$$

$$\sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) \le \sum_{i=0}^{\log_b n} c^i f(n) \le f(n)\sum_{i=0}^{\infty} c^i = f(n)\frac{1}{1-c}$$

Hence

$$T(n) \le f(n)\frac{1}{1-c} + O(n^{\log_b a}) = O(f(n))$$

Improvement: Matrix-Matrix Multiplication

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Divide and Conquer Let

$$\begin{aligned}
M_1 &= (A_{11} + A_{22})(B_{11} + B_22) \\
M_2 &= (A_{21} + A_{22})B_{11} \\
M_3 &= A_{11}(B_{12} - B_{22}) \\
M_4 &= A_{22}(B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12})B_{22} \\
M_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}$$

Strassen Algorithm: Matrix-Matrix Multiplication

$$AB = \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix}$$

Then the recurrence relation is

$$T(n) = 7T\left(\frac{n}{2}\right) + O(n^2)$$

Comparing it with Master theorem, $a = 7, b = 2$ and $f(n) = O(n^2)$. Therefore,

$$T(n) = O(n^{\log_b a}) = O(n^{\log_2 7}) = O(n^{2.8074})$$

Improvement: Matrix-Matrix Multiplication

- Strassen Algorithm is due to Strassen [1969], fast matrix multiplication, used in practice.
- The latest algorithm is due to Williams, Xu, Xu and Zhou [2024][1] that we can do matrix-matrix multiplication by $O(n^{2.371552})$, but they are called galactic algorithm, not used in practice.
- Remember that Strassen algorithm is not galactic.

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[1] Vassilevska Williams, Virginia; Xu, Yinzhan; Xu, Zixuan; Zhou, Renfei. New Bounds for Matrix Multiplication: from Alpha to Omega. Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). pp. 3792–3835. arXiv:2307.07970. doi:10.1137/1.9781611977912.134.

Order and Rate of Convergence

Order and Rate of Convergence

Definition 2 (Linear Convergence and Rate of Convergence)
Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers converging to a limit L. We say that the convergence is at least linear if there exists a constant $0 < c < 1$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = c$$

The number c is called the rate of convergence.

Example: 2^{-n}

Superlinear Convergence

Definition 3 (Superlinear)
Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers converging to a limit L. We say that the convergence is superlinear if there exists a sequence $r_n \to 0$ and an integer N such that

$$|x_{n+1} - L| \leq r_n |x_n - L|, n \geq N$$

Equivalently

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = 0$$

Example: n^{-n}

Sublinear Convergence

Definition 4 (Sublinear)
Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers converging to a limit L. We say that the convergence is sublinear if

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = 1$$

Example: $\frac{1}{n}$

logarithmic Convergence

Definition 5 (Logarithmic)

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers converging to a limit L. We say that the convergence is logarithmic if

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = 1$$

and

$$\lim_{n\to\infty} \frac{|x_{n+2} - x_{n+1}|}{|x_{n+1} - x_n|} = 1$$

Example: $\frac{1}{n}$

Order of Convergence

Definition 6 (Order and Rate of Convergence)

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers converging to a limit L. The sequence is said to converge with order $\alpha > 1$ to L and with a rate of convergence c if

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|^{\alpha}} = c$$

$$\text{where } c \in \begin{cases} (0, \infty) & \text{if } \alpha > 1 \\ (0, 1) & \text{if } \alpha = 1 \end{cases}$$

If $\alpha = 1$, linearly convergent, $\alpha = 2$, quadratically convergent, $\alpha = 3$, cubically convergent and so on.

Order of Convergence

Example 7

Let $\{a_n\}$ be a positive sequence converging to 0. Find the order and rate of convergence (Find α and c) if

$$\begin{aligned}(1)a_{n+1} &= a_n^2\\(2)a_{n+1} &= a_n(1-a_n)\\(3)a_n &= n^{-n}\end{aligned}$$

Answer: (1) $\alpha = 2, c = 1$ (quadratic), (2) $\alpha = 1, c = 1$ (sublinear), (3) $\alpha = 1, c = 0$ (superlinear)

Exercise

Exercise 1: Medium

12. Prove that $\frac{1}{2^n}$ converges linearly with rate of convergence $\frac{1}{2}$
13. Prove that $\frac{1}{2^{2^n}}$ converges superlinearly or quadratically
14. Prove that $\frac{1}{2^{n^2}}$ converges superlinearly.
15. Prove that $\frac{1}{n+1}$ converges logarithmically or sublinearly.
16. Prove that $\frac{1}{n^k}, k > 0$ converges linearly.

Exercise

Exercise 2: Hard

16. Prove that

$$\frac{n^\alpha}{n+1} \to \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

Thanks

Doubts and Suggestions

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