MA633L-Numerical Analysis

Lecture 41: Numerical Differentiation - Multi-Step Method

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Multistep Methods

Multistep Method

Definition 1 (m-step multistep method)

An **m-step multistep method** for solving the IVP (**??**) has a difference function for finding the approximation $y(t_{i+1})$ represented by the following equation, where m is an integer greater than 1:

$$
\begin{aligned}
y(t_{i+1}) &= \sum_{j=0}^{m-1} a_j y(t_{i-(m-j-1)}) + h\left[\sum_{j=0}^{m-1} b_j f(t_{i-(m-j-1)}, y(t_{i-(m-j-1)}))\right] \\
&\quad + h b_m f(t_{i+1}, y(t_{i+1}))
\end{aligned}
$$

for $i = m-1, m, \cdots, n-1$, where $h = (T - t_0)/n$, $a_j's$ and $b_j's$ are constants and $y_0, y(t_1), y(t_2) \cdots, y(t_{m-1})$ are specified.

Multistep Method

$$
\begin{aligned}
y(t_{i+1}) = a_{m-1}y_i + a_{m-2}y_{i-1} + \cdots + a_0 y_{i+1-m} \\
+ h[b_{m-1}f(t_i, y_i) + b_{m-2}f(t_{i-1}, y_{i-1}) \\
+ \cdots + b_0 f(t_{i+1-m}, y_{i+1-m})] \\
+ hb_m f(t_{i+1}, y(t_{i+1}))
\end{aligned}
$$

When $b_m = 0$, the method is called explicit or open as $y(t_{i+1})$ is written explicitly in terms of previous values. When $b_m \neq 0$, it is called an implicit method or closed as $y(t_{i+1})$ occurs on both sides of the equation.

Multistep Method

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)dt = \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

$$\implies y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

Since f is continuous, we can use a corresponding interpolating polynomial $P(t)$ to $f(t, y(t))$ using data points

$(t_i, f(t, y(t_i))), (t_{i-1}, f(t_{i-1}, y(t_{i-1}))), \cdots, (t_{i-(m-1)}, f(t_{i-(m-1)}, y(t_{i-(m-1)})))$.

Since $P(t)$ is the interpolating polynomial of $f(t, y(t))$, we have

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} P(t)dt$$

Newton backward or forward difference for an interpolating polynomial.

Adams-Bashforth explicit m-step methods - backward difference polynomial $P_{m-1}(t)$.

Multistep Method

The multistep method can be written as

$$
\begin{aligned}
y(t_{i+1}) = y(t_i) + h \bigg[& f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i)) \\
& + \frac{3}{8}\nabla^3 f(t_i, y(t_i)) + \frac{251}{720}\nabla^4 f(t_i, y(t_i)) \\
& + \frac{95}{288}\nabla^5 f(t_i, y(t_i)) \bigg] + K h^{m+1} f^{(m)}(\xi_i, y(\xi_i))
\end{aligned}
\tag{1}
$$

Adams-Bashforth method

Adams-Bashforth method

It is an explicit method. Therefore, $y_0, y_1, \cdots y_{m-1}$ must be specified. Further, $b_{m-1}, \cdots, b_0$ must be specified. $a_{m-1} = 1$ and $a_0 = a_1 = \cdots = a_{m-2} = 0 = b_m$.

Adams-Bashforth Two-Step Explicit Method

Specify y_0 and y_1, then compute . Let us derive the two-step Adams-Bashforth method from equation (1). Here $m = 2$. Therefore we have

$$
\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]\right] \\
&= y(t_i) + \frac{h}{2}\left[3f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))\right]
\end{aligned}
$$

$$
\boxed{y(t_{i+1}) = y(t_i) + \frac{h}{2}[3f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]} \qquad (2)
$$

where $i = 1, 2, \cdots, n-1$. The error is $\frac{5}{12}y'''(\xi_i)h^2$ for some $\xi_i \in (t_{i-1}, t_{i+1})$.

Adams-Bashforth Three-Step Explicit Method

Specify y_0, y_1, and y_2, then compute. Let us derive the three-step Adams-Bashforth method from equation (1) Here $m = 3$. Therefore we have

$$
\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]\right. \\
&\left. + \frac{5}{12}[f(t_i, y(t_i)) - 2f(t_{i-1}, y(t_{i-1})) + f(t_{i-2}, y(t_{i-2}))]\right] \\
&= y(t_i) + \frac{h}{12}\left[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1}))\right. \\
&\left. + 5f(t_{i-2}, y(t_{i-2}))\right]
\end{aligned}
$$

Adams-Bashforth Three-Step Explicit Method

$$y(t_{i+1}) = y(t_i) + \frac{h}{12}[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) \tag{3}$$

$$+ 5f(t_{i-2}, y(t_{i-2}))] \tag{4}$$

where $i = 2, 3 \cdots, n-1$. The error is $\frac{3}{8}y^{(4)}(\xi_i)h^3$ for some $\xi_i \in (t_{i-2}, t_{i+1})$.

Adams-Bashforth Four-Step Explicit Method

Specify y_0, y_1, y_2 and y_3, then we can derive the following from (1)

$$y(t_{i+1}) = y(t_i) + \frac{h}{24}[55f(t_i, y(t_i)) - 59f(t_{i-1}, y(t_{i-1})) \tag{5}$$

$$+ 37f(t_{i-2}, y(t_{i-2})) - 9f(t_{i-3}, y(t_{i-3}))] \tag{6}$$

where $i = 3, 4, \cdots, n-1$. The error is $\frac{251}{720}y^{(5)}(\xi_i)h^4$ for some $\xi_i \in (t_{i-3}, t_{i+1})$.

Adams-Bashforth Five-Step Explicit Method

Specify y_0, y_1, y_2, y_3 and y_4, then we can derive the following equation from (1).

$$
\begin{aligned}
y(t_{i+1}) = y(t_i) + \frac{h}{720}[1901 f(t_i, y(t_i)) - 2774 f(t_{i-1}, y(t_{i-1})) \\
+ 2616 f(t_{i-2}, y(t_{i-2})) - 1274 f(t_{i-3}, y(t_{i-3})) \\
+ 251 f(t_{i-4}, y(t_{i-4}))]
\end{aligned}
$$

where $i = 4, 5, \cdots, n-1$. The error is $\frac{95}{288} y^{(6)}(\xi_i) h^4$ for some $\xi_i \in (t_{i-4}, t_{i+1})$.

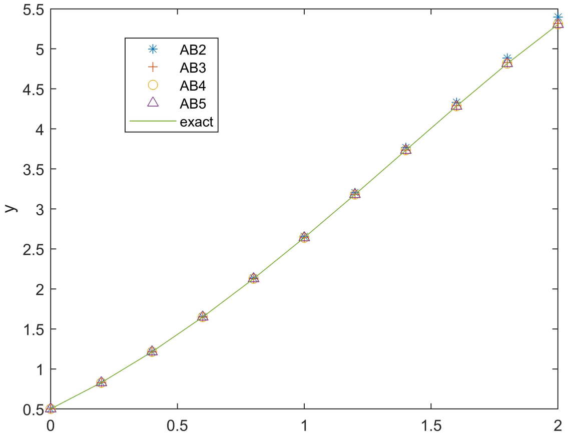
Adams-Bashforth

Example 2

Using the Adams-Bashforth two-, three-, four- and five-step explicit methods, solving the following IVP: $y' = 1 - t^2 + y \quad 0 \leq t \leq 2, y(0) = 0.5.$ use the following information appropriately $y(0.2) = 0.8293, y(0.4) = 1.2141, y(0.6) = 1.6489, y(0.8) = 2.1272$ with $n = 10.$

Adams-Bashforth

The following table shows Adams-Bashforth two-, three-, four-, and five-step explicit methods' approximations and errors



Adams-Bashforth

t_i	$y_{exact}(t_i)$	$AB2(t_i)$	E_t	$AB3(t_i)$	E_t	$AB4(t_i)$	E_t	$AB5(t_i)$	E_t
0	0.5000	0.5000	0	0.5000	0	0.5000	0	0.5000	0
0.2000	0.8293	0.8293	0.0000	0.8293	0.0000	0.8293	0.0000	0.8293	0.0000
0.4000	1.2141	1.2161	0.0020	1.2141	0.0000	1.2141	0.0000	1.2141	0.0000
0.6000	1.6489	1.6540	0.0050	1.6494	0.0004	1.6489	0.0000	1.6489	0.0000
0.8000	2.1272	2.1366	0.0093	2.1283	0.0011	2.1272	0.0000	2.1272	0.0000
1.0000	2.6409	2.6561	0.0153	2.6428	0.0020	2.6410	0.0002	2.6409	0.0000
1.2000	3.1799	3.2033	0.0234	3.1831	0.0032	3.1803	0.0003	3.1800	0.0000
1.4000	3.7324	3.7667	0.0343	3.7373	0.0049	3.7330	0.0006	3.7325	0.0001
1.6000	4.2835	4.3324	0.0489	4.2906	0.0071	4.2844	0.0009	4.2836	0.0001
1.8000	4.8152	4.8835	0.0683	4.8254	0.0102	4.8165	0.0013	4.8154	0.0002
2.0000	5.3055	5.3992	0.0938	5.3197	0.0142	5.3074	0.0019	5.3057	0.0003

Adams-Moulton method

Adams-Moulton technique

Adams-Moulton method is obtained by using an additional interpolation node $(t_{i+1}, f(t_{i+1}, y(t_{i+1})))$ in the approximation of the integral

$$\int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

It is an implicit method. We have to specify $y_0, y_1, y_2, \cdots, y_{m-1}$. Further $b_m, b_{m-1}, \cdots, b_0$ must be specified. $a_{m-1} = 1$ and $a_0 = a_1 = \cdots = a_{m-2} = 0$.

Adams-Moulton One-Step Implicit Method

Here, y_0 must be specified. Employing the trapezoidal rule for the integration produces the following result. Here $m = 1$. Therefore we have

$$\begin{aligned} y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i))\right] \\ &= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_{i+1}, y(t_{i+1})) - f(t_i, y(t_i))]\right] \\ &= y(t_i) + \frac{h}{2}[f(t_{i+1}, y(t_{i+1})) + f(t_i, y(t_i))] \end{aligned}$$

$$\boxed{y(t_{i+1}) = y(t_i) + \frac{h}{2}[f(t_{i+1}, y(t_{i+1})) + f(t_i, y(t_i))]} \qquad (7)$$

where $i = 0, 1, 2, \cdots, n-1$. The error is $-\frac{1}{24}y^{(4)}(\xi_i)h^3$ for some $\xi_i \in (t_{i-1}, t_{i+1})$.

Adams-Moulton Two-Step Implicit Method

Here, y_0 and y_1 must be specified. Here $m = 2$. Therefore we have

$$
\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}[f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]\right. \\
&\left.+ \frac{5}{12}[f(t_{i+1}, y(t_{i+1})) - 2f(t_i, y(t_i)) + f(t_{i-1}, y(t_{i-1}))]\right] \\
&= y(t_i) + \frac{h}{12}\left[5f(t_{i+1}, y(t_{i+1})) + 8f(t_i, y(t_i)) - 5f(t_{i-1}, y(t_{i-1}))\right]
\end{aligned}
$$

$$
\boxed{y(t_{i+1}) = y(t_i) + \frac{h}{12}[5f(t_{i+1}, y(t_{i+1})) + 8f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))]} \qquad (8)
$$

where $i = 1, 2, \cdots, n-1$. The error is $-\frac{1}{24}y^{(4)}(\xi_i)h^3$ for some $\xi_i \in (t_{i-1}, t_{i+1})$.

Adams-Moulton Three-Step Implicit Method

We must specify y_0, y_1 and y_2 and the formula is

$$y(t_{i+1}) = y(t_i) + \frac{h}{24}[9f(t_{i+1}, y(t_{i+1})) + 19f(t_i, y(t_i)) \\ - 5f(t_{i-1}, y(t_{i-1})) + f(t_{i-2}, y(t_{i-2}))]$$

where $i = 2, 3, \cdots, n-1$. The error is $-\frac{19}{720}y^{(5)}(\xi_i)h^4$ for some $\xi_i \in (t_{i-2}, t_{i+1})$.

Adams-Moulton Four-Step Implicit Method

We must specify y_0, y_1, y_2 and y_3 and the formula is

$$y(t_{i+1}) = y(t_i) + \frac{h}{720}[251f(t_{i+1}, y(t_{i+1})) + 646f(t_i, y(t_i)) \\ - 264f(t_{i-1}, y(t_{i-1})) + 106f(t_{i-2}, y(t_{i-2})) - 19f(t_{i-3}, y(t_{i-3}))]$$

where $i = 3, 4, \cdots, n-1$. The error is $-\frac{3}{160}y^{(6)}(\xi_i)h^5$ for some $\xi_i \in (t_{i-3}, t_{i+1})$.

BVP

BVP

- Similar to IVP.
- A BVP has conditions specified at the boundaries of the independent variables.

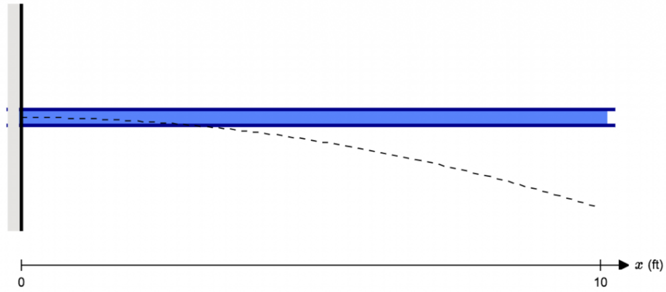
BVP

Deflection of Beam

$$\frac{d^2w}{dx^2} = \frac{s}{EI}w(x) + \frac{q(x)}{2EI}(x-l) \tag{9}$$

No deflection at the ends of the beams

$$w(0) = 0, \quad w(l) = 0$$



BVP

The second-order differential equation

$$\frac{d^2y}{dx^2} = f(x, y(x), y'(x)), a \leq x \leq b \tag{10}$$

together with two boundary conditions

$$y(a) = \alpha, y(b) = \beta \tag{11}$$

It is called the two-point boundary variable problem.

BVP or IVP

Classify as BVP and IVP

1. $\frac{d^2y}{dt^2} = -2t + e^t, t \geq 0, \;\; y(0) = 0, y'(0) = 2$
2. $\frac{d^2y}{dx^2} = 4x + 5y, 0 \leq x \leq 1, \;\; y(0) = 4, y(1) = 5$
3. $\frac{d^3y}{dx^3} = xy + x^3e^x, 0 \leq x \leq 1, \;\; y(0) = y(1) = 0, y'(0) = 1$
4. $\frac{d^3y}{dx^3} = xy + (x^3 - 2x^2)e^x, x \geq 0, \;\; y(0) = y'(0) = 0, y''(0) = 1$
5. $\frac{d^2y}{dx^2} + 4y = 0, 0 \leq x \leq \frac{\pi}{2}, \;\; y(0) = -2, y(\frac{\pi}{4}) = 10$

Boundary Conditions

A linear combination of y and y' at the boundary points

$$a_1 y(a) + b_1 y'(a) = \alpha$$

$$a_2 y(b) + b_2 y'(b) = \beta$$

- **Dirichlet**: $a_1, a_2 \neq 0$ and $b_1, b_2 = 0$
- **Neumann**: $a_1, a_2 = 0$ and $b_1, b_2 \neq 0$
- **Robin**: $a_1, a_2, b_1, b_2 \neq 0$
- **Mixed**: $a_1 \neq 0, b_2 \neq 0, a_2 = 0, b_1 = 0$ or $a_2 \neq 0, b_1 \neq 0, a_1 = 0, b_2 = 0$
- **Cauchy**: $y(a) = \alpha, y'(a) = \beta$.

BCs

Dirichlet Boundary Conditions

$\frac{d^2y}{dx^2} = 4x + 5y, 0 \leq x \leq 1, \;\; y(0) = 4, y(1) = 5$

- Essential
- Fixed
- First Type

Neumann Boundary Conditions

$$\frac{d^2y}{dx^2} = 4x + 5y, 0 \leq x \leq 1, \;\; y'(0) = 4, y'(1) = 5$$

- Natural

Robin Boundary Conditions

$\frac{d^2y}{dx^2} = 4x + 5y, 0 \leq x \leq 1, \;\; 2y(0) + 3y'(1) = 4, 4y(0) + 8y'(1) = 5$

Mixed Boundary Conditions

$$\frac{d^2y}{dx^2} = 4x + 5y, 0 \leq x \leq 1, \;\; y(0) = 4, y'(1) = 5$$

Cauchy Boundary Conditions

$$\frac{d^2y}{dx^2} = 4x + 5y, 0 \le x \le 1, \;\; y(0) = 4, y'(0) = 5$$

Boundary Conditions

In general, if we have a differential equation defined on a domain $\Omega \subset \mathbb{R}^n$ and its boundary conditions prescribed at the boundary of the domain $\partial\Omega$, it is called a boundary value problem.

BVP

Theorem 3

Suppose the function f in the boundary-value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, y(b) = \beta$$

is continuous on the set

$$\Omega = \{(x, y, y') : a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous on Ω. If

1. $f_y(x, y, y') > 0$ for all $(x, y, y') \in \Omega$ and
2. there exists a constant M such that

$$|f_{y'}(x, y, y')| \le M \quad \text{for all } (x, y, y') \in \Omega$$

then the BVP has a unique solution.

BVP

Show that the BVP

$$y'' + e^{-xy} + \sin y' = 0, \quad 1 \le x \le 2, \quad y(1) = y(2) = 0$$

has a unique solution.

Solution 4

We have

$$f(x, y, y') = -e^{-xy} - \sin y', \quad \forall x \in [1, 2]$$

Also,

$$f_y(x, y, y') = xe^{-xy} > 0$$

and

$$|f_{y'}(x, y, y')| = |-\cos y'| \le 1$$

Therefore, it satisfies all conditions of the above theorem, and hence it has a unique solution.

Linear BVP

Linear BVP

The differential equation

$$y'' = f(x, y, y')$$

is linear when there exists functions $p(x), q(x)$ and $r(x)$ such that

$$f(x, y, y') = p(x)y' + q(x)y + r(x)$$

Linear BVP

Theorem 5

Suppose linear BVP

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, y(b) = \beta$$

satisfies

1. $p(x), q(x), r(x) \in C[a, b]$
2. $q(x) > 0$ Then the BVP has a unique solution.

Linear IVP

Theorem 6

If the initial value problem

$$y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b, \quad y(a) = \alpha, y'(a) = \beta$$

satisfies

1. $p(x), q(x), r(x) \in C[a, b]$
2. $q(x) > 0$ Then the IVP has a unique solution.

Thanks

Doubts and Suggestions

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MA633L-Numerical Analysis

Lecture 41: Numerical Differentiation - Multi-Step Method

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