भारतीय प्रौद्योगिकी संस्थान तिरुपति

TIRUPATI

MA633L-Numerical Analysis
Numerical ODEs-
Finite Difference Method

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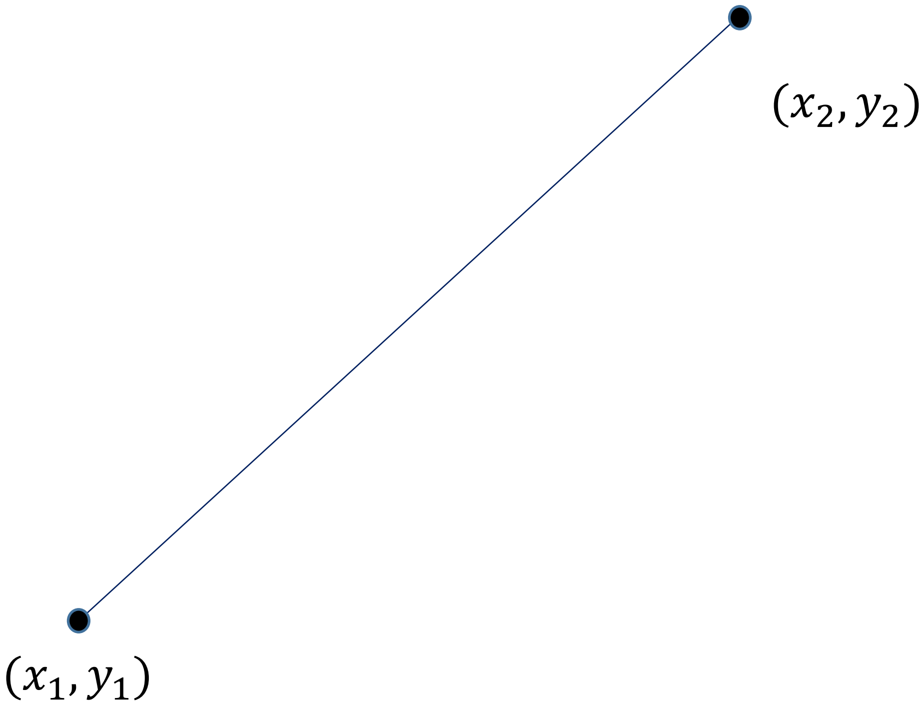
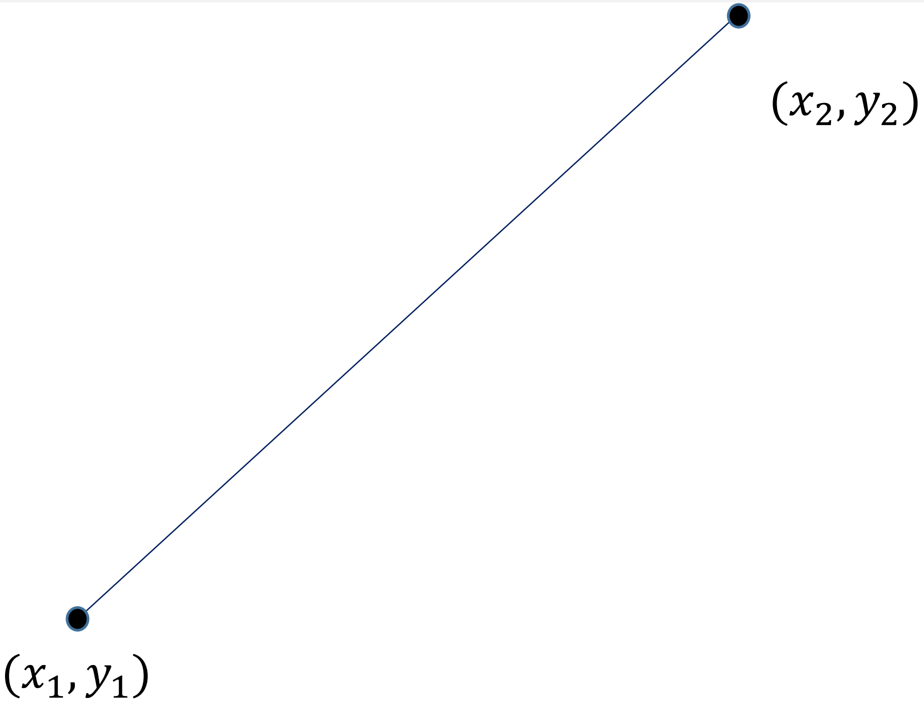
IIT Tirupati**

Taylor Series and Numerical Analysis

Straight Line

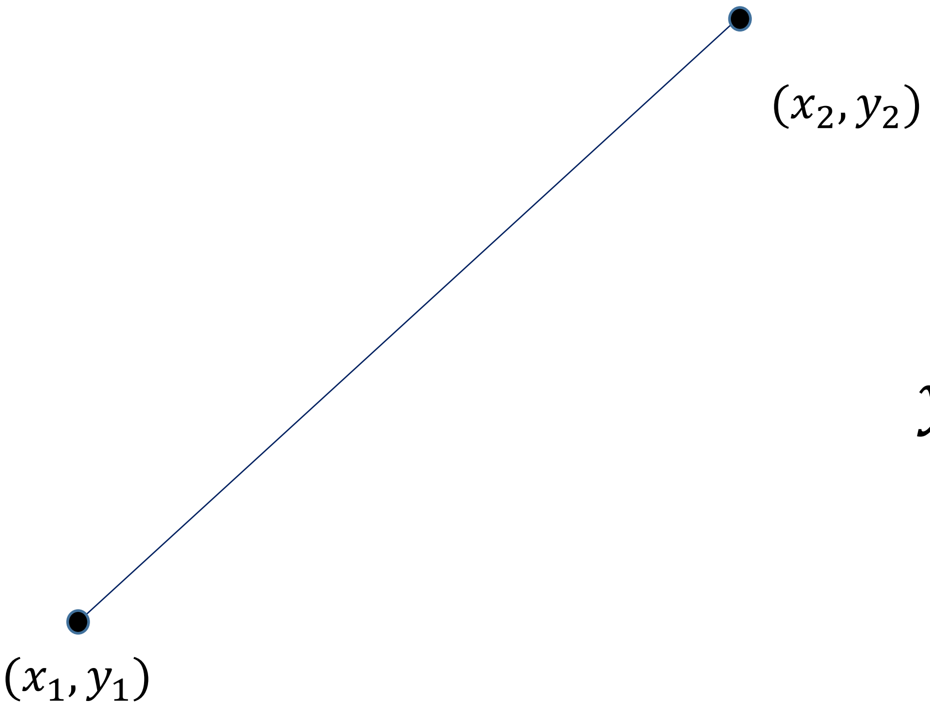


Straight Line



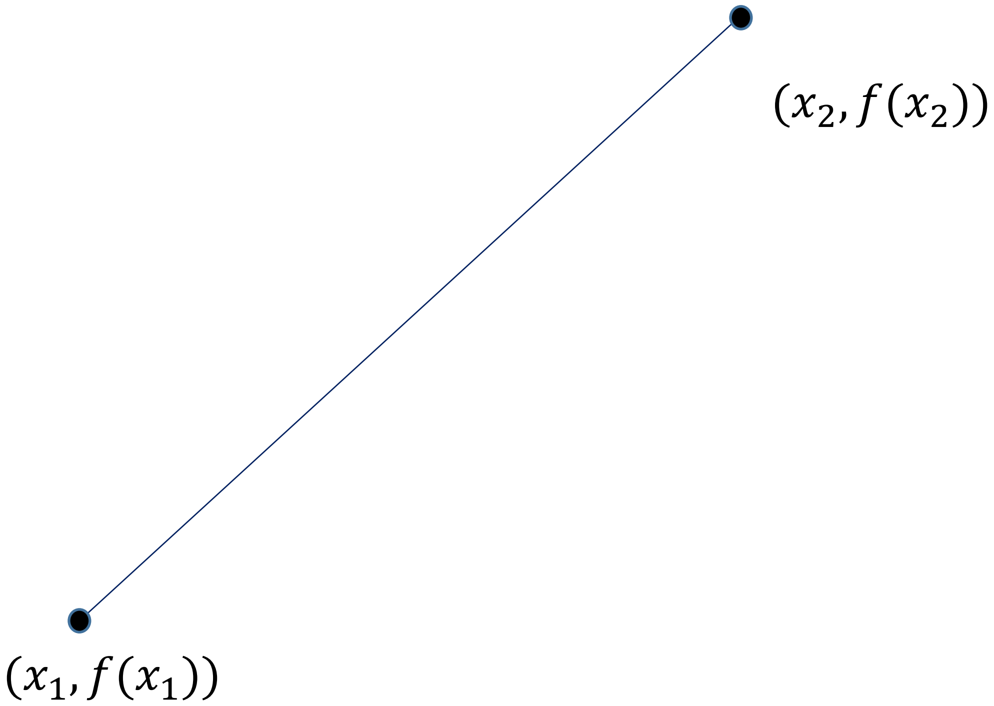
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Straight Line



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Straight Line



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

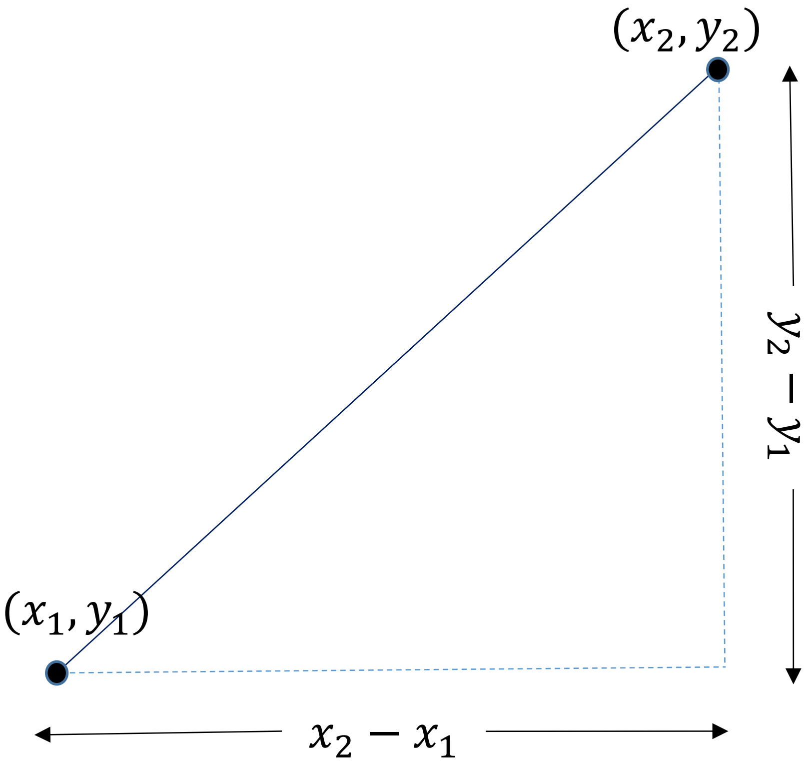
$$y = f(x)$$

$$y_1 = f(x_1)$$

$$y_2 = f(x_2)$$

$$f(x) = f(x_1) + \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(x - x_1)$$

Slope

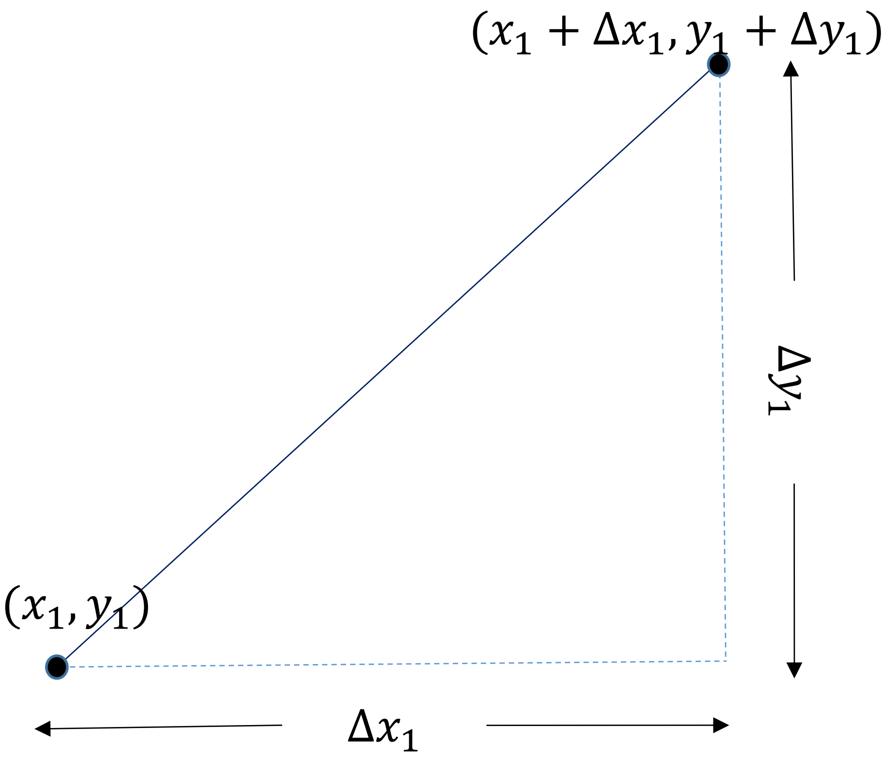


$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y = y_1 + m(x - x_1)$$

Slope

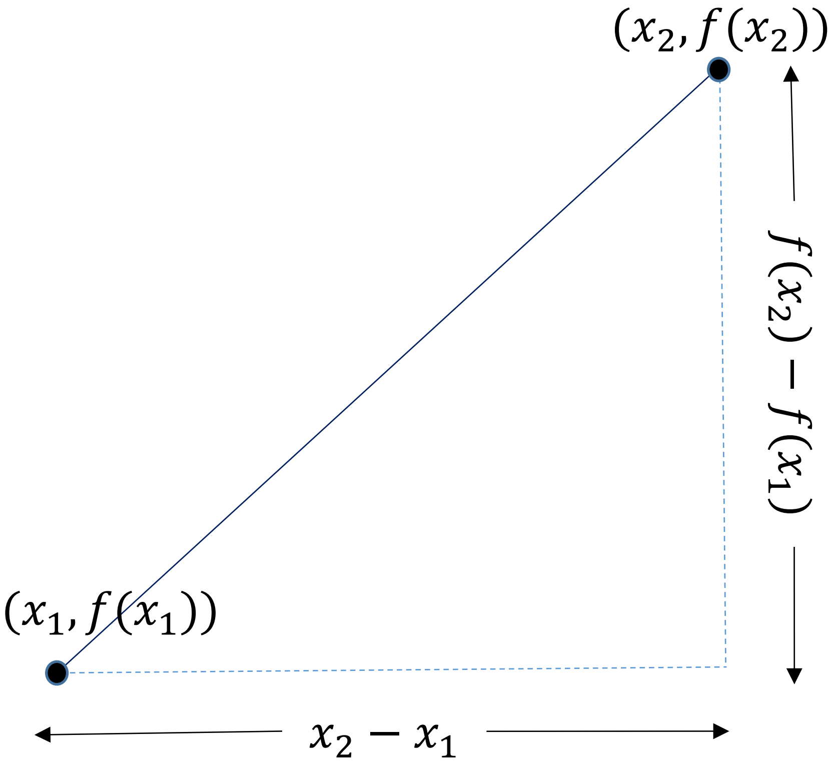


$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$m = \frac{\Delta y_1}{\Delta x_1}$$

$$y = y_1 + m(x - x_1)$$

Slope



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

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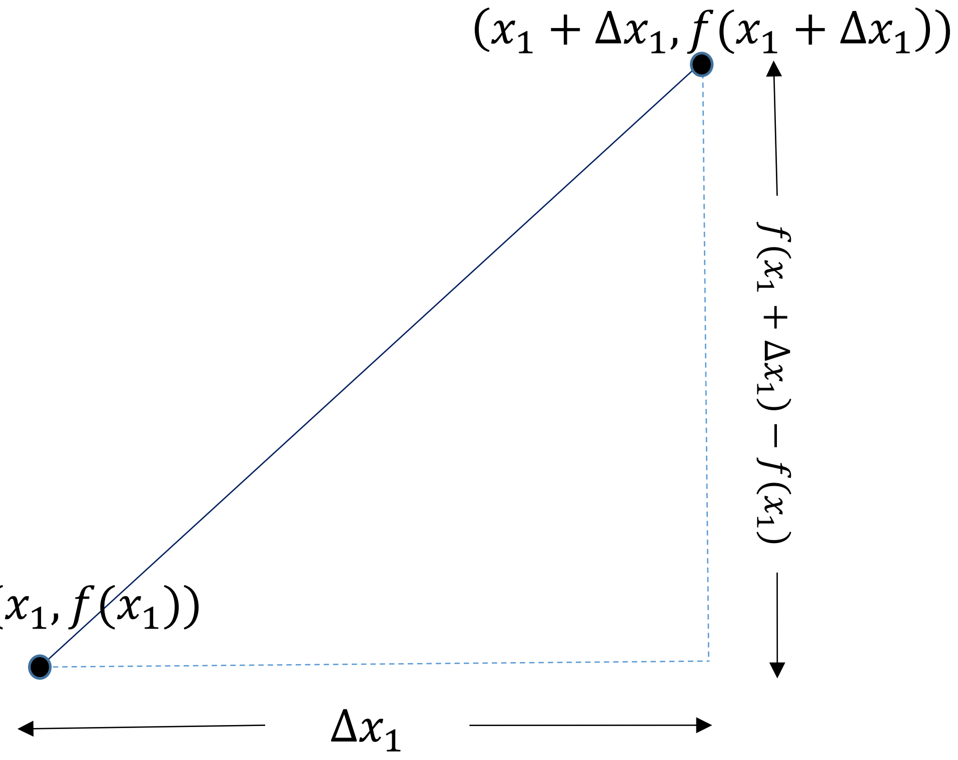
$$y_1 = f(x_1)$$

$$y_2 = f(x_2)$$

$$f(x) = f(x_1) + \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(x - x_1)$$

Slope



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

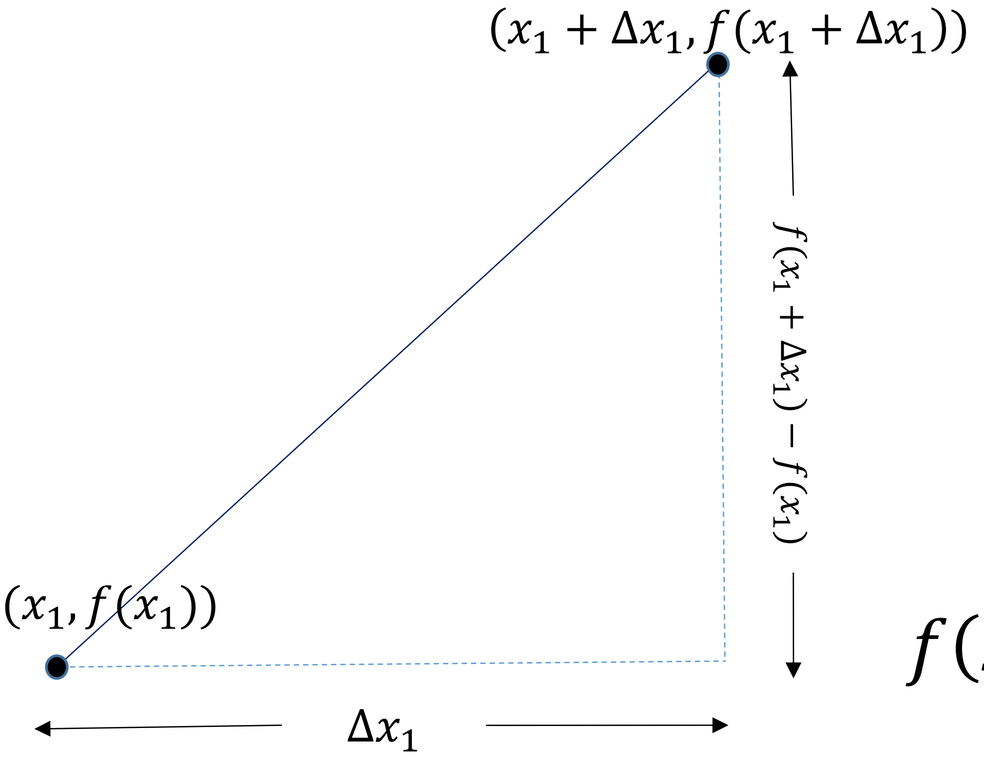
$$y = f(x)$$

$$y_1 = f(x_1)$$

$$y_2 = f(x_1 + \Delta x_1)$$

$$f(x) = f(x_1) + \left(\frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1}\right)(x - x_1)$$

Slope



$$y = f(x)$$

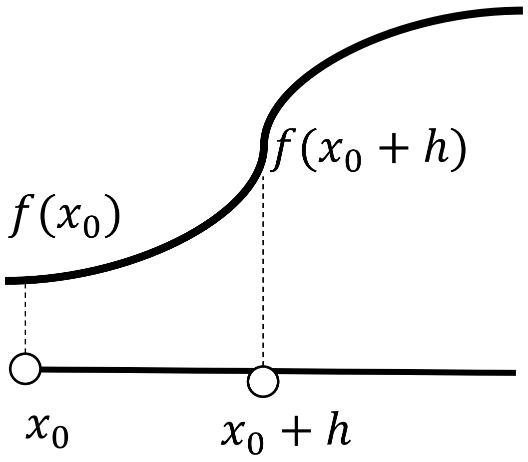
$$f(x) = f(x_1) + \left(\frac{dy}{dx}\right)_{x=x_1} (x - x_1)$$

$$f(x) = f(x_1) + f'(x_1)(x - x_1)$$

Taylor's Theorem

$$f(x_0 + h) = \sum_{n}^{N} \frac{f^{(n)}(x_0)h^n}{n!} + R_N(x)$$

Taylor's Approximation



$$f(x_0 + h) = f(x_0) + f'(x_0)h + R_1(x)$$

$$f(x_0 + h) - f(x_0) = f'(x_0)h + R_1(x)$$

Dividing by h

$$\frac{f(x_0 + h)}{h} - \frac{f(x_0)}{h} = f'(x_0) + \frac{R_1(x)}{h}$$

Taylor's Approximation

$$\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h} = f'(x_0) + \frac{R_1(x)}{h}$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{R_1(x)}{h}$$

Assume $\frac{R_1(x)}{h}$ *is small, then*

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

Forward Euler Scheme

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\frac{dy}{dt} = g(t, y)$$

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + \Delta t g_n$$

Backward Euler Scheme

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\frac{dy}{dt} = f(t, y)$$

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + \Delta t f_{n+1}$$

Euler Schemes

Explicit Euler Method

$$y_{n+1} = y_n + \Delta t f_n$$

Implicit Euler Method

$$y_{n+1} = y_n + \Delta t f_{n+1}$$

Euler Schemes

$$y' = -ky, y(0) = y_0$$

Explicit Euler Method

$$y_{exact} = y_0 e^{-kt}$$

$$y_{n+1} = y_n - \Delta t k y_n$$

$$y_{n+1} = (1 - \Delta t k) y_n$$

Implicit Euler Method

$$y_{n+1} = y_n - \Delta t k y_{n+1}$$

$$(1 + \Delta t k) y_{n+1} = y_n$$

$$y_{n+1} = \frac{y_n}{(1 + \Delta t k)}$$

Centered Difference

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\frac{dy}{dt} = f(t, y)$$

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + \Delta t f_{n+\frac{1}{2}}$$

Taylor's Approximation to solve ODE

Let us solve the following ODE

$$u' = 3u + 2$$

From Taylor's approximation,

$$u'(x_0) \approx \frac{u(x_0 + h) - u(x_0)}{h}$$

$$u' = 3u + 2$$

$$\frac{u(x_0 + h) - u(x_0)}{h} = 3u(x_0) + 2$$

Taylor's Approximation to solve ODE

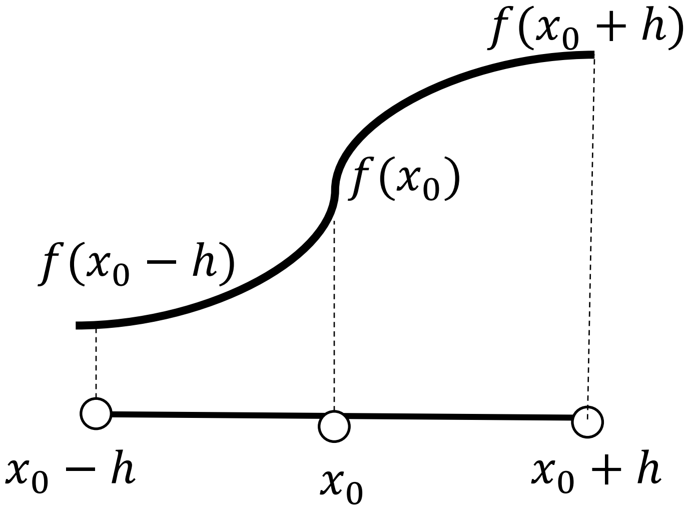
$$u' = 3u + 2$$

$$\frac{u(x_0 + h) - u(x_0)}{h} = 3u(x_0) + 2$$

$$u(x_0 + h) - u(x_0) = 3hu(x_0) + 2h$$

$$u(x_0 + h) = u(x_0) + 3hu(x_0) + 2h$$

Taylor's Approximation 2nd Order



$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2} + R_2(x)$$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)h^2}{2} + R_2^*(x)$$

$$f(x_0+h) + f(x_0-h) \approx 2f(x_0) + f''(x_0)h^2$$

$$f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

Taylor's Approximation to solve ODE

Let us solve the following ODE

$$u'' = 3u + 2$$

From Taylor's approximation,

$$u''(x_0) \approx \frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2}$$

$$u'' = 3u + 2$$

$$\frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2} = 3u(x_0) + 2$$

Taylor's Approximation to solve ODE

$$\frac{u(x_0+h)-2u(x_0)+u(x_0-h)}{h^2}=3u(x_0)+2$$

Assume $u(x_0)=u_i, u(x_0-h)=u_{i-1},$

$$u(x_0+h)=u_{i+1}$$

Then

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}=3u_i+2$$

$$u_{i+1}-2u_i+u_{i-1}=3h^2u_i+2h^2$$

$$u_{i-1}-(2+3h^2)u_i+u_{i+1}=2h^2$$

Taylor's Approximation to solve ODE

$$u_{i-1} - (2+3h^2)u_i + u_{i+1} = 2h^2$$

i=1

$$(2+3h^2)u_1 + u_2 = 2h^2$$

i=2

$$u_1 + (2+3h^2)u_2 + u_3 = 2h^2$$

i=3

$$u_2 + (2+3h^2)u_3 + u_4 = 2h^2$$

i=N

$$u_{N-1} + (2+3h^2)u_N = 2h^2$$

Taylor's Approximation to solve ODE

$$(2+3h^2)u_1 + u_2 = 2h^2$$
$$u_1 + (2+3h^2)u_2 + u_3 = 2h^2$$
$$u_2 + (2+3h^2)u_3 + u_4 = 2h^2$$
$$u_{N-1} + (2+3h^2)u_N = 2h^2$$
$$Au = b$$
$$a = 2 + 3h^2$$

$$A = \begin{bmatrix} a & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & a \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2h^2 \\ 2h^2 \\ \vdots \\ 2h^2 \end{bmatrix}$$

Taylor Series on two Variables

$$z = z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right)(x - x_1) + \left(\frac{z_2 - z_1}{y_2 - y_1}\right)(y - y_1)$$

$$z = f(x, y)$$

$$f(x,y) = f(x_1, y_1) + \left(\frac{f(x_1 + \Delta x_1, y_1) - f(x_1, y_1)}{\Delta x_1}\right)(x - x_1) + \left(\frac{f(x_1, y_1 + \Delta y_1) - f(x_1, y_1)}{\Delta y_1}\right)(y - y_1)$$

Taylor Series on two Variables

$$z = z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right)(x - x_1) + \left(\frac{z_2 - z_1}{y_2 - y_1}\right)(y - y_1)$$

$$z = f(x, y)$$

$$f(x, y) = f(x_1, y_1) + \left(\frac{\partial f(x_1, y_1)}{\partial x}\right)(x - x_1) + \left(\frac{\partial f(x_1, y_1)}{\partial y}\right)(y - y_1)$$

Taylor Series on two Variables

$$f(x,y) = f(x_1,y_1) + \left(\frac{\partial f(x_1,y_1)}{\partial x}\right)(x - x_1) + \left(\frac{\partial f(x_1,y_1)}{\partial y}\right)(y - y_1) + \frac{1}{2!}\left[\left(\frac{\partial^2 f(x_1,y_1)}{\partial x^2}\right)(x - x_1)^2 + \left(\frac{\partial^2 f(x_1,y_1)}{\partial y^2}\right)(x - x_1)(y - y_1) + \left(\frac{\partial^2 f(x_1,y_1)}{\partial y^2}\right)(y - y_1)^2\right]$$

Taylor Series

- f is infinitely differentiable about a point x_0

$$f(x) = \sum_{n}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Taylor Series on two Variables

$$f(x_1, x_2, \dots, x_d) = \sum_{n_1}^{\infty} \dots \sum_{n_d}^{\infty} \frac{(x_1 - x_0^1)^{n_1} \dots \left(x_d - x_0^d\right)^{n_d}}{n_1! \dots n_d!} \left(\frac{\partial^{n_1 + \dots + n_d} f}{\partial x_1^{n_1} \dots \partial x_d^{n_d}} \right) (x_0^1, \dots, x_0^d)$$

$$f(x, y) = \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \frac{(x - x_0)^{n_1} (y - y_0)^{n_2}}{n_1! \; n_2!} \left(\frac{\partial^{n_1 + n_2} f}{\partial x^{n_1} \partial y^{n_2}} \right) (x_0, y_0)$$

1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \begin{matrix} 0 < x < l, \\ 0 < t < \infty \end{matrix}$$

$$T(0, t) = f(t), 0 < t < \infty$$

$$T(l, t) = g(t), 0 < t < \infty$$

$$T(x, 0) = s(x), 0 \leq x \leq l$$

Diffusion:
behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle.

α-diffusion coefficient

1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < l, \quad 0 < t < \infty$$

$$T(0,t) = f(t), 0 < t < \infty$$

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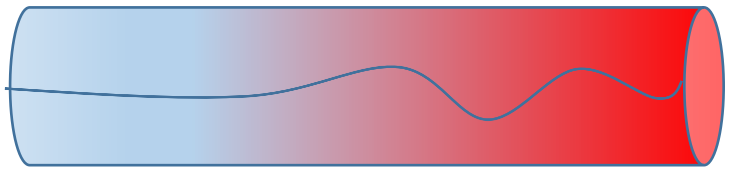


1D Heat Equation

- **Diffusion equation converges to a stationary solution $T_s(x)$ as $t \to \infty$**
- **In this limit $T_t = 0$ and $T_s''(x) = 0$**
- **This stationary limit of the diffusion equation is called Laplace equation**

1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad \begin{matrix} 0 < x < l, \\ 0 < t < \infty \end{matrix}$$

$$T(0,t) = 0, 0 < t < \infty$$

$$T(l,t) = 0, 0 < t < \infty$$

$$T(x,0) = s(x), 0 \leq x \leq l$$

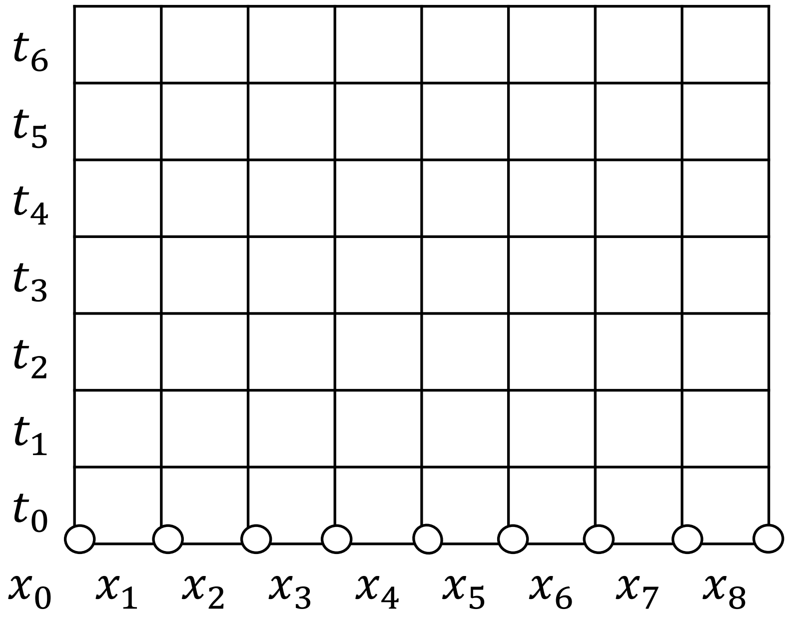
Forward Euler Scheme

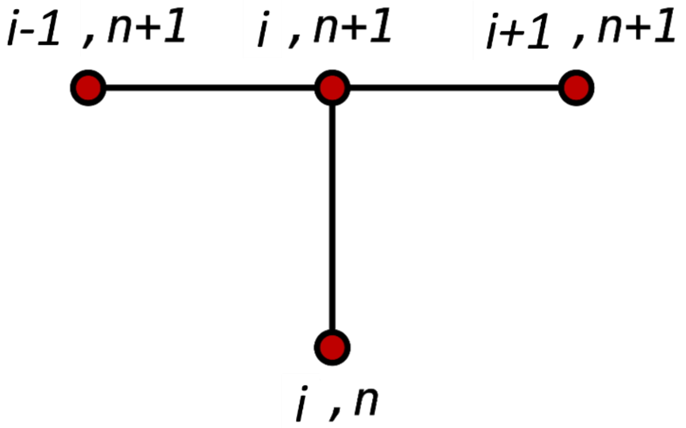
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$x_i = i\Delta x, i = 0,1,2, \dots, N_x$$

$$t_n = n\Delta t, n = 0,1,2, \dots, N_t$$

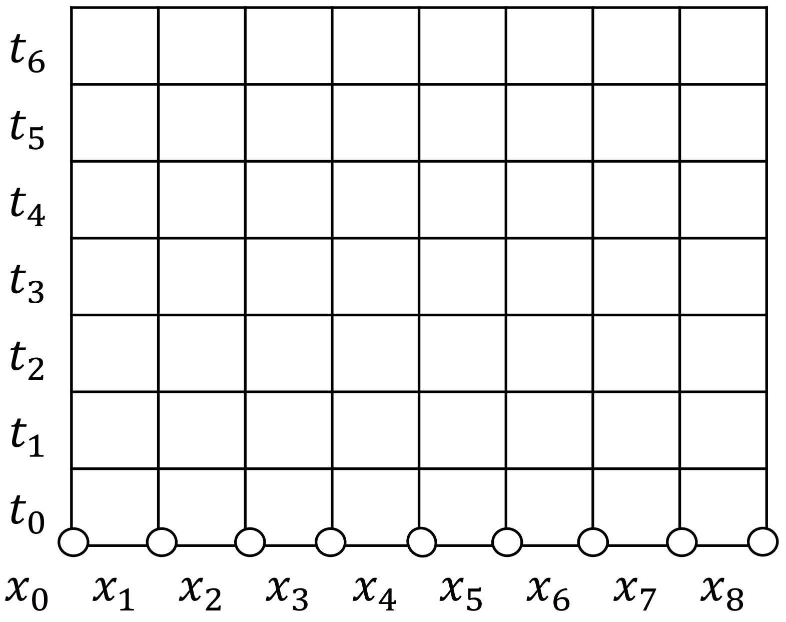
$$T(x_i, t_n) = T_i^n$$





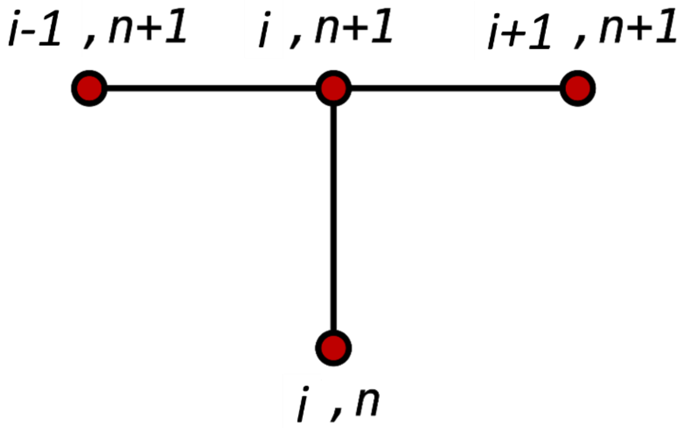
Forward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$



$$T(x_i, t_n) = T_i^n$$

$$\frac{\partial T(x_i, t_n)}{\partial t} = \alpha \frac{\partial^2 T(x_i, t_n)}{\partial x^2} + f(x_i, t_n)$$



$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

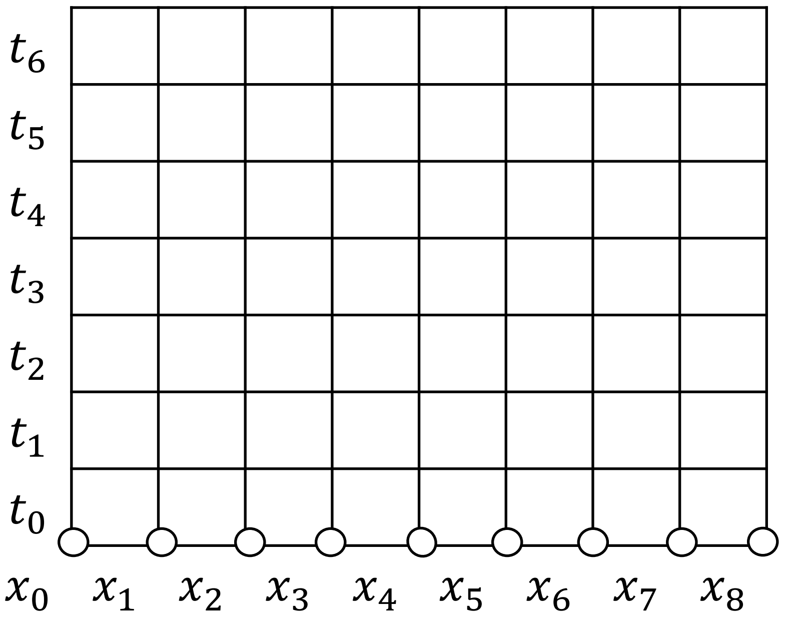
Forward Euler Scheme

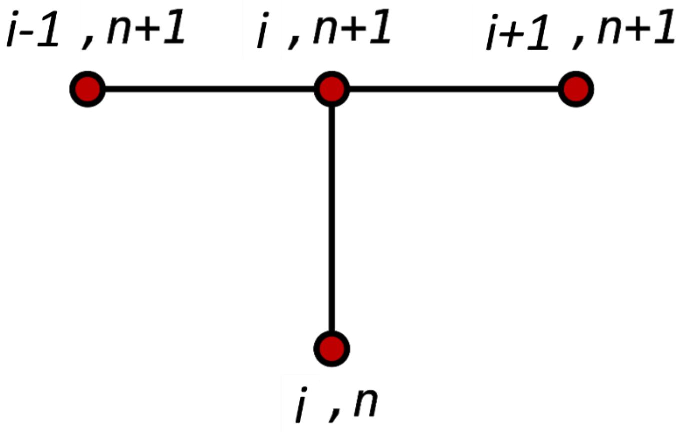
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$T(x_i, t_n) = T_i^n$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

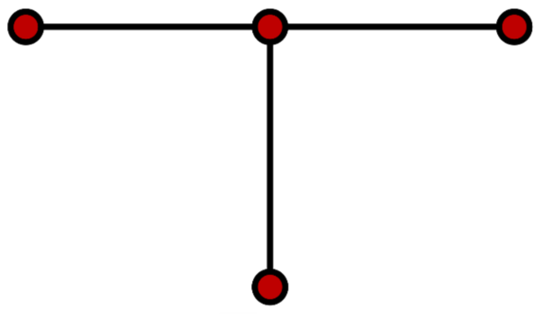
$$T_i^{n+1} = T_i^n + \alpha \frac{\Delta t}{\Delta x^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + f_i^n \Delta t$$



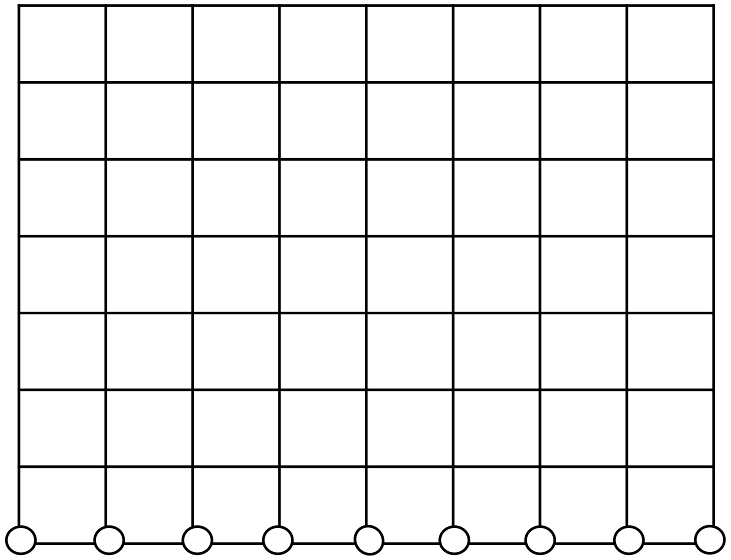


Forward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$



$$T(x_i, t_n) = T_i^n$$



$$T_i^{n+1} = T_i^n + F(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + f_i^n \Delta t$$

$$F = \alpha \frac{\Delta t}{\Delta x^2}$$

F is the dimensionless number that lumps the key physical parameter in the problem, α, and the discretization parameters Δx and Δt into a single parameter. Depending on F, the numerical method schemes are chosen.

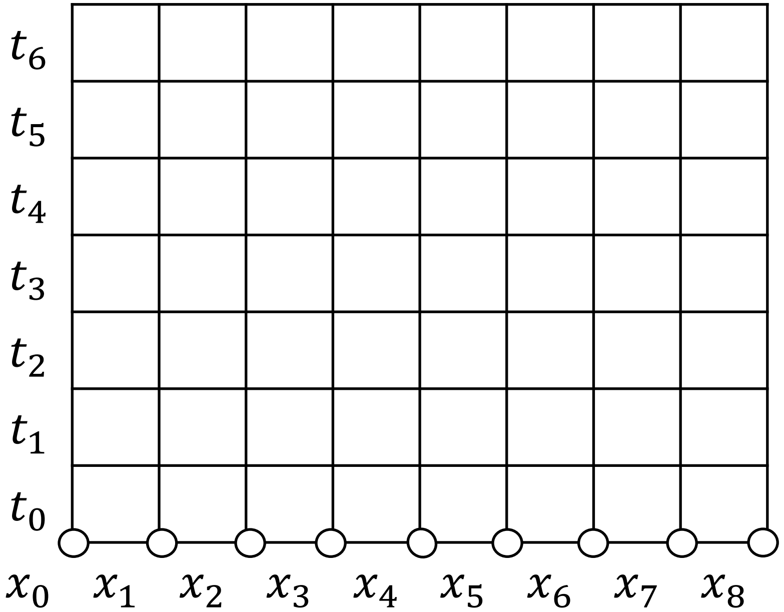
Backward Euler Scheme

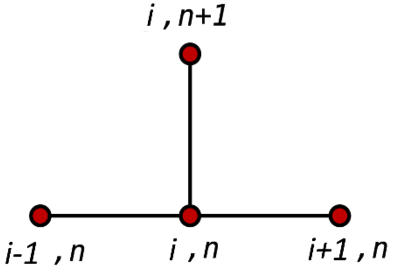
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$x_i = i\Delta x, i = 0,1,2, \dots , N_x$$

$$t_n = n\Delta t, n = 0,1,2, \dots , N_t$$

$$T(x_i, t_n) = T_i^n$$

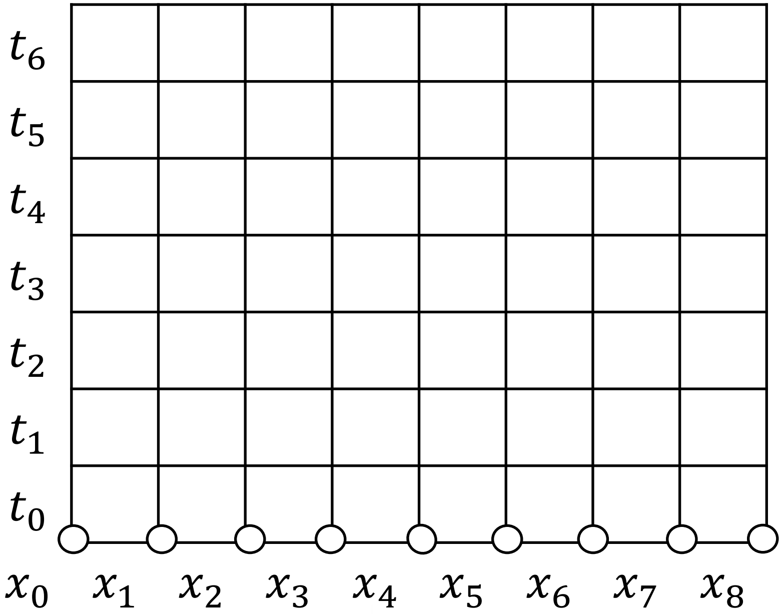




Backward Euler Scheme

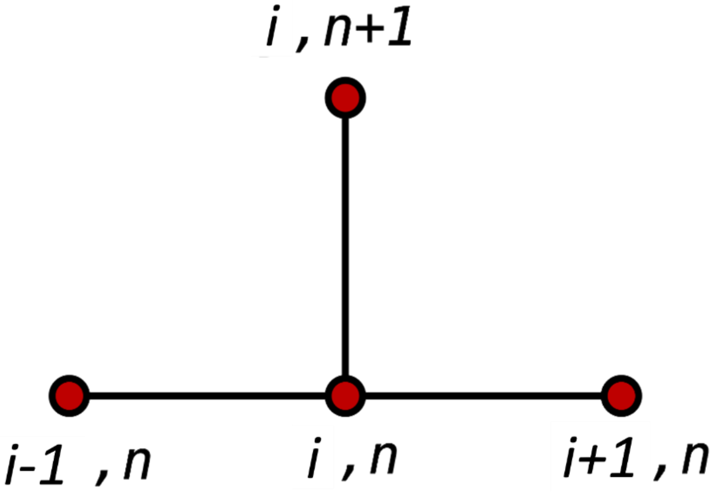
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$f''(x_0) \approx \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$



$$T(x_i, t_n) = T_i^n$$

$$\frac{\partial T(x_i, t_n)}{\partial t} = \alpha \frac{\partial^2 T(x_i, t_n)}{\partial x^2} + f(x_i, t_n)$$



$$\frac{T_i^n - T_i^{n-1}}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

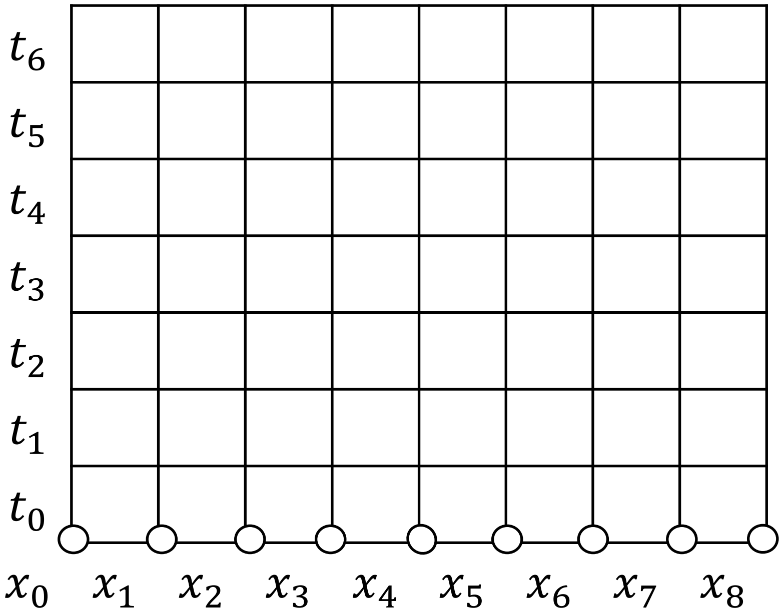
Backward Euler Scheme

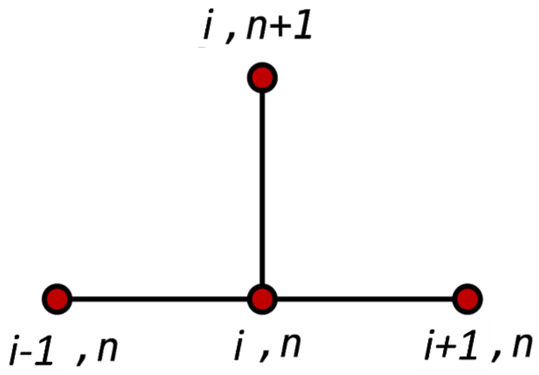
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$

$$\frac{T_i^n - T_i^{n-1}}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

$$T_i^n - T_i^{n-1} = \alpha\left(\frac{\Delta t}{\Delta x^2}\right)(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t f_i^n$$

$$T_i^n - T_i^{n-1} = F(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t f_i^n$$

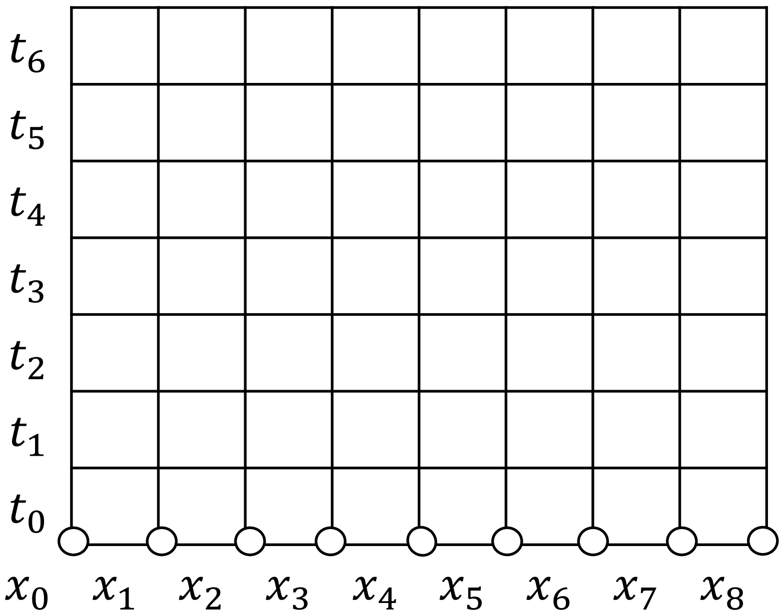




Backward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$T(x_i, t_n) = T_i^n$$



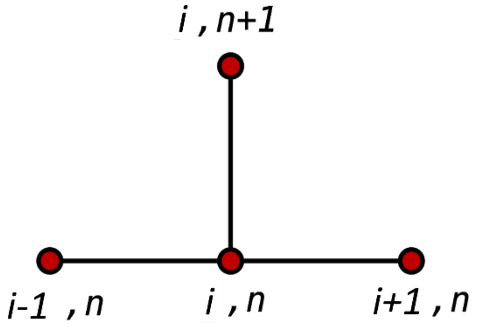
$$T_i^n - T_i^{n-1} = F(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t f_i^n$$

$$-FT_{i-1}^n + (1 + 2F)T_i^n - FT_{i+1}^n = T_{i-1}^{n-1} + \Delta t f_i^n$$

$$-FT_0^n + (1 + 2F)T_1^n - FT_2^n = T_1^{n-1} + \Delta t f_1^n$$

$$-FT_1^n + (1 + 2F)T_2^n - FT_3^n = T_2^{n-1} + \Delta t f_2^n$$

$$-FT_2^n + (1 + 2F)T_3^n - FT_4^n = T_3^{n-1} + \Delta t f_3^n$$



Backward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$

$$-FT_{i-1}^n + (1+2F)T_i^n - FT_{i+1}^n = T_{i-1}^{n-1} + \Delta t f_i^n$$

$$AT = b$$

$$A = \begin{bmatrix} a_{00} & a_{01} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{10} & a_{11} & a_{12} & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & a_{21} & a_{22} & a_{23} & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 0 & a_{ii-1} & a_{ii} & a_{ii+1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{N_x-1N_x} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_{N_xN_x-1} & a_{N_xN_x} \end{bmatrix}$$

$$a_{ii-1} = a_{ii+1} = -F$$

$$a_{ii} = 1 + 2F$$

1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$0 < x < l,$$
$$0 < t < \infty$$

$$T(0,t) = 0, 0 < t < \infty$$

$$T(l,t) = 0, 0 < t < \infty$$

$$T(x,0) = s(x), 0 \leq x \leq l$$

$T(x,0) = s(x)$

$T(0,t) = f(t)$

$T(l,t) = g(t)$

Backward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$

$$-FT_{i-1}^n + (1+2F)T_i^n - FT_{i+1}^n = T_{i-1}^{n-1} + \Delta t f_i^n$$

$$AT^n = b \qquad a_{ii-1} = a_{ii+1} = -F \qquad a_{ii} = 1 + 2F$$

$$A = \begin{bmatrix} a_{00} & a_{01} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{10} & a_{11} & a_{12} & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & a_{21} & a_{22} & a_{23} & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 0 & a_{ii-1} & a_{ii} & a_{ii+1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{N_x-1N_x} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_{N_xN_x-1} & a_{N_xN_x} \end{bmatrix}$$

$$b_0 = b_{N_x} = 0, b_i = T_i^{n-1}$$

$$T^n = \begin{bmatrix} T_0^n \\ T_1^n \\ T_2^n \\ \vdots \\ T_{N_x-1}^n \\ T_{N_x}^n \end{bmatrix} \qquad b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N_x-1} \\ b_{N_x} \end{bmatrix}$$

End of Lecture