MA633L-Numerical Analysis

Lecture 5 : Error Stories

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TIRUPATI

Master Theorem: Recap

Master Theorem

Theorem 1

Let $a > 0, b > 1$ be constants and let $f(n)$ be a driving function that is defined and nonnegative on all sufficiently large reals. Define the recurrence relation $T(n), n \in \mathbb{N}$ by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant k, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if f satisfies the smoothness condition $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Continuous Master Theorem

Theorem 2

Let $a > 0, b > 1$ be constants and let $f(n)$ be a driving function that is defined and nonnegative on all sufficiently large reals. Define the recurrence relation $T(n), n \in \mathbb{R}$ by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Then

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant k, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if f satisfies the smoothness condition $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Order of Convergence

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|^{\alpha}} = c, \text{ where } c \in \begin{cases} (0, \infty) & \text{if } \alpha > 1 \\ (0, 1) & \text{if } \alpha = 1 \end{cases}$$

- $\alpha = 1$, linear
- $\alpha = 2$, quadratic
- $\alpha = 3$, cubic
- $\alpha = 1, c = 0$, superlinear
- $\alpha = 1, c = 1$, sublinear
- $\alpha = 1, \lim_{n\to\infty} \frac{|x_{n+2} - x_{n+1}|}{|x_{n+1} - x_n|} = 1$, logarithmic

Errors: Real ~~Ho~~Error Stories

Story 1: Mars Climate Orbiter Failure

Example 3

In 1998, NASA launched the Mars Climate Orbiter with the objective of studying the Martian climate and weather patterns.

Mission: To enter orbit around Mars and send back valuable data about the planet's atmosphere. Precise calculations were required to ensure that the spacecraft would enter the correct orbit around Mars.

Mission Failed.

Link : https://science.nasa.gov/mission/mars-climate-orbiter/

Story 1: Mars Climate Orbiter Failure

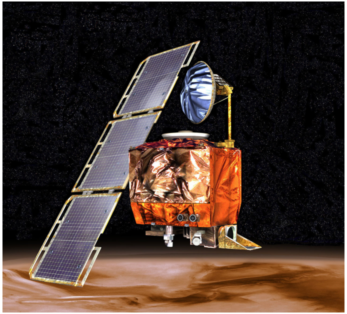

Figure 1: Mars Climate Orbiter, Source: Wikipedia

Story 1: Mars Climate Orbiter Failure

Numerical Error: Failure due to metric conversion mistakes. The Jet Propulsion laboratory (JPL) engineers used meters and kilograms in their spacecraft's thrust calculations whereas Lockheed Martian Astronautics who designed and built the spacecraft provided the data in feet. Therefore, the acceleration calculation in pounds-seconds^2 was not translated properly to Newtons-seconds^2. When the probe came too close to mars and maneuver into orbit and destroyed. Later, it was found that it was due to failed translation of the English units into metric units in a piece of software.
Financial Loss: \$327 million

Historical Story 2: The Vasa Warship: Failure

Example 4

This warship sank less than a mile into here maiden voyage and 30 people died in 1628. Later it was found that it is due to different systems of measurement. Two rulers were used Swedish feet = 12 inches and another two rulers used Amsterdam feet = 11 inches.

Construction and Design

- Commissioned by King Gustavus Adolphus, Sweden, powerful warship
- To carry large number of cannons and had ornate carvings, decorations
- Two gun decks, heavily armed

Link: https://www.vasamuseet.se/en

Historical Story 2: The Vasa Warship: Failure

The Maiden Voyage:

- August 10, 1628, From Stockholm harbor on it maiden voyage
- Traveled only 1300 meters, strong gust of wind
- The ship heeled (tilted) to port, righted itself slightly, and then heeled again. Water began rushing in through the open gun ports, causing the Vasa to capsize and sink.

Design Flaws:

- Center of gravity was too high, unstable
- Design changed multiple times, lack of experienced ship builders
- No adequate testing for stability test

Historical Story 2: The Vasa Warship: Failure

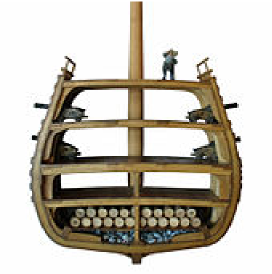

Figure 2: Vasa Warship Hull, Source: https://wiki.kidzsearch.com/wiki/Vasa_%28ship%29

Historical Story 2: The Vasa Warship: Failure

Construction Flaws:

- Hull was asymmetrical
- Port side (left): Measurement from keel to deck were found to be slightly different from seaboard side (right)
- Seaboard side (right): The height from the keel to the deck was shorter by about an inch compared to the port side.

Story 3: The Gimli Glider Descent and Landing

Another story on unit conversion

Example 5

When Air Canada flight 143 (between Montreal and Edmonton) used its first craft in metric measurements, the pilot measured fuel tank in dripstick, converted the measurement of volume into one of weight in pounds and entered in the management computer. Therefore, less than half the amount of intended fuel had been loaded which have made one engine failure. However, the pilot was able to glide the aircraft safely at an Air Force base in Gimli.

Story 3: The Gimli Glider Descent and Landing

Figure 3: Air Canada Flight After Landing in Gimli, Source: https://admiralcloudberg.medium.com/a-mathematical-miracle-the-story-of-air-canada-flight-143-or-the-gimli-glider-9e99545d9b3d

Story 3: Laufenburg Bridge

Another story on unit conversion in 1990s

Example 6

It is a bridge to connect the towns of Laufenburg in Switzerland and Laufenburg in Germany, which are separated by the Rhine River at the cost of $12 million. Sea levels varies from place to place. Germany uses its sea level reference from Amsterdam. Switzerland takes its sea-level reference from Geneva. Two sea levels reference, one from North Sea and another from Mediterannean. has a height difference as 27cm. Builders made miscalculations and one side was 54cm higher than the other.

Mistakes were corrected and then adjustments made to build and bridge was open to public in 2003.

Story 3: Laufenburg Bridge

Figure 4: Laufenburg Bridge, Source: Wikipedia

Story 4: The 2018 Boeing 737 MAX Crashes

Example 7

- Lion Air Flight 610: On October 29, 2018, Lion Air Flight 610 crashed into the Java Sea shortly after takeoff from Jakarta, Indonesia. All 189 passengers and crew members were killed.
- Ethiopian Airlines Flight 302: On March 10, 2019, Ethiopian Airlines Flight 302 crashed near Addis Ababa, Ethiopia, shortly after takeoff. All 157 people on board perished.

A new software was added for automatic control system named as Maneuvering Characteristics Augmentation System (MCAS) Boeing flights. An aerodynamics stall occurs when the aircraft exceeds its critical angle of attack. Due to wrong angle of attack calculations, Indonesia flight crash and Ethiopia flight crash occured.

Story 4: The 2018 Boeing 737 MAX Crashes

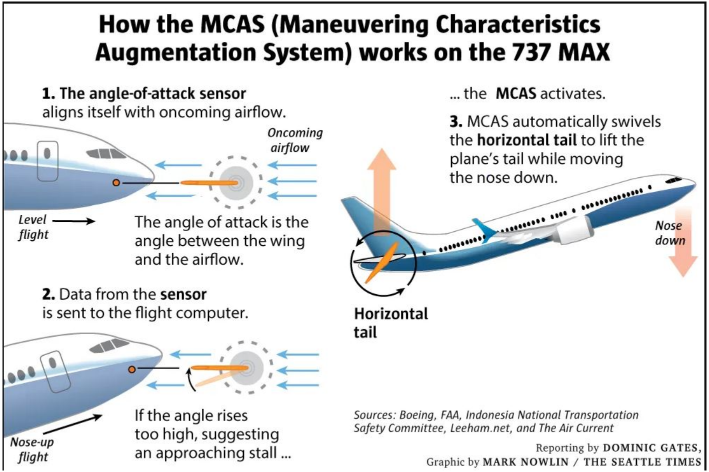

Figure 5: MCAS Bridge, Source: Link

Story 4: The 2018 Boeing 737 MAX Crashes

- MCAS was designed to automatically push the aircraft’s nose down if the system detected a potential stall, based on inputs from angle-of-attack (AOA) sensors.
- MCAS system relied on data from AOA sensors to make decisions. Faulty AOA sensor readings led the MCAS to erroneously believe the aircraft was in a stall condition, causing it to repeatedly push the nose of the aircraft down

The grounding of the 737 MAX had severe financial repercussions for Boeing, including legal settlements and a decline in orders for the aircraft.
For more details: See this Video

Story 5: The Patriot Missile Failure

Example 8

American Patriot Missile battery in Saudi Arabia during Gulf war failed to attack Iraqi scud missile and made human deaths. The cause was an inaccurate calculation of the time since boot due to computer arithmetic errors. The time in tenths of second as measured by the internal clock was multiplied by 10 to produce the time in seconds. This calculation was performed in 24 bit fixed point register. In particular, the value 1/10, which has a non-terminating binary expansion, was chopped at 24 bits after the radix point. The small chopping error, when multiplied by the large number giving the time in tenths of a second, led to a significant error.

Story 5: The Patriot Missile Failure

Indeed, the Patriot battery had been up around 100 hours, and an easy calculation shows that the resulting time error due to the magnified chopping error was about 0.34 seconds. (The number 1/10 equals 1/24+1/25+1/28+1/29+1/212+1/213+.... In other words, the binary expansion of 1/10 is 0.0001100110011001100110011001100.... Now the 24 bit register in the Patriot stored instead 0.00011001100110011001100 introducing an error of 0.0000000000000000000000011001100... binary, or about 0.000000095 decimal. Multiplying by the number of tenths of a second in 100 hours gives $0.000000095 \times 100 \times 60 \times 60 \times 10 = 0.34.$).

Story 5: The Patriot Missile Failure

A Scud travels at about 1,676 meters per second, and so travels more than half a kilometer in this time. This was far enough that the incoming Scud was outside the "range gate" that the Patriot tracked. More details can be found here.

For video details: See this video

Story 6: Ariane 5 Rocket Disaster

Example 9

On June 4, 1996 an unmanned Ariane 5 rocket launched by the European Space Agency exploded just forty seconds after its lift-off from Kourou, French Guiana. Ariane explosion The rocket was on its first voyage, after a decade of development costing $7 billion. The destroyed rocket and its cargo were valued at $500 million. A board of inquiry investigated the causes of the explosion and in two weeks issued a report. It turned out that the cause of the failure was a software error in the inertial reference system. Specifically a 64 bit floating point number relating to the horizontal velocity of the rocket with respect to the platform was converted to a 16 bit signed integer. The number was larger than 32,767, the largest integer storeable in a 16 bit signed integer, and thus the conversion failed. More details can be found here.

Story 7: Parking Garage Tickets: Decimals and Time

Example 10

A traffic warden new to the job discovered in 2008 when he incorrectly ticketed several cars at a parking lot in Devon, England. In a parking lot, drivers would pay in advance for a certain number of minutes and would then put a ticket in the windshield showing the time they had entered the lot and how many minutes of parking they had paid for. One driver, Dave Alsop, entered at 2:49 pm and paid for 75 minutes. This meant that the car was covered until 4:04 pm.

Story 7: Parking Garage Tickets: Decimals and Time

The traffic warden, however, determined the expiration time by entering 14.49 into his calculator (for 1449 in 24-hour time, which corresponds to 2:49 pm) and adding on 0.75 (for the 75 minutes). The traffic warden got 15.24 as the sum, and he interpreted that number to mean that the car was only covered until 3:24 pm. It was just past that time, so the warden issued a penalty. When Mr Alsop returned to the car at 3:41 pm, he saw the £50 fine and found the traffic warden. Mr Alsop said that he tried to explain the error — that hours have 60 minutes, not 100, so standard decimal addition does not apply — but the warden did not see any problem. Three cars incorrectly received fines due to this error. For more details see here.

Accuracy and Precision

Errors

- Inaccuracy or bias is defined as systematic deviation from the truth
- Imprecision or uncertainty refers to magnitude of the scatter
- When a numerical method is designed, it should be sufficiently accurate or unbiased to meet the requirements of the particular problem
- There should be precise enough for adequate design

Errors

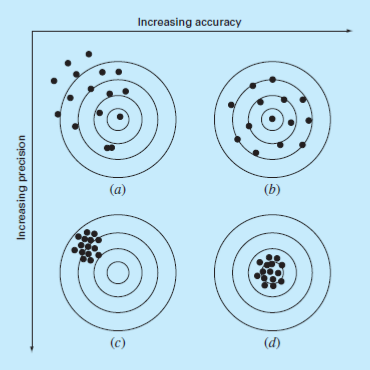

Figure 6: Accuracy and Precision, (a) inaccurate and imprecise, (b) accurate and imprecise, (c) inaccurate and precise (d) accurate and precise. Source: S. Chapra, R. P. Canale, Numerical Methods for Engineers

Error Definitions

Error Definitions

$$\text{True Value} = \text{Approximation} + \text{True Error} \tag{1}$$

Let E_t denote the true error or exact value of the error or absolute error, then we obtain

$$\text{True Error } (E_t) = \text{True Value} - \text{Approximation} \tag{2}$$

Error Definitions

The disadvantage of this definition is that, it is not taking into account the order of magnitude of the value. Alternatively, you can normalize it and define True fractional relative error or true percentage relative error

$$\text{True fractional relative error}(\varepsilon_t) = \frac{\text{true value} - \text{approximation}}{\text{true value}} \tag{3}$$

$$\text{true percentage relative error }(\varepsilon_t\%) = \frac{\text{true value} - \text{approximation}}{\text{true value}} 100\% \tag{4}$$

Error Definitions

Let us use the following definitions

Definition 11 (True Error, Absolute Error and Relative Error)
Suppose that $\tilde{a}$ is an approximation to the true value a. Then the true error or actual error is

$$E_t = a - \tilde{a},$$

the absolute error is

$$E_{tabs} = |a - \tilde{a}|$$

and the relative error is

$$\varepsilon_t = \frac{|a - \tilde{a}|}{|a|},$$

provided that $a \neq 0$.

Problem

Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (a) E_t and ε_t for each case.
Answer: $E_t = 1$ cm and $\varepsilon_t = 0.01\%$ for bridge and $E_t = 1$ cm and $\varepsilon_t = 10\%$ for rivet. The relative error for the rivet is much greater.

Error Definitions

In practical, true value is rarely available, for example, in numerical methods true values are known only when analytical solution or true answer is known a priori. An alternative to normalize the error using the best available estimate of the true value, that is the approximation itself,

$$\text{percent relative error } (\varepsilon_a\%) = \frac{\text{present approx.} - \text{previous approx.}}{\text{present approx.}} 100\% \quad (5)$$

Error Definitions

Definition 12 (Relative Error)
Suppose that $\tilde{a}$ is the present approximation to the previous approximation a. The relative error is

$$\varepsilon_a = \frac{|a - \tilde{a}|}{|a|},$$

provided that $a \neq 0$.

Error Definitions

Often, we may not concern the sign of the error, in computations, but interested in the absolute value of the percent relative error is lower than prescribed tolerance ε_s. We repeat the computation until, it satisfies the following stopping criterion.

$$|\text{percent relative error}| < \varepsilon_s \tag{6}$$

If

$$\varepsilon_s = (0.5 \times 10^{2-n})\%$$

it is assured the result is correct to at least n significant figures.

For our course, unless specified, $n = 8$.

Thanks

Doubts and Suggestions

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