MA633L-Numerical Analysis

Lecture 8 : Numerical Interpolation-Introduction

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Basic Idea

Basic Idea

Consider the following NIRF Score and Rank Data

Institute	Score	Rank
IITM	89.46	1
IITB	86.66	2
IITKGP	76.88	5
IITI	64.72	16
IITPKD	50.24	64

Suppose IIT Tirupati has the score of 50.53, what will be its rank?

Basic Idea

Consider the following marks of students and their respective grades for the Numerical Analysis course.

Student	Marks	Grade
Student 1	88	S
Student 2	68	B
Student 3	75	A
Student 4	30	E
Student 5	55	C

Suppose you have scored 45 marks in this course. What could be your grade?

Basic Idea

- Assume that you have designed a geographical website which shows the temperature of various cities based on the longitude and latitude. The following table represents the temperature from measurement device located at the respective weather monitor

City	**Latitude** $^\circ N$	**Longitude** $^\circ E$	**Temperature** ($^\circ C$)
Tirupati	13.6288	79.4192	27
Kalahasti	13.7527	79.7067	25
Venkatagiri	13.9591	79.5808	26
Chittoor	13.4788	79.8383	27
Vellore	12.9236	79.1331	27

What could be the temperature of IIT Tirupati whose coordinates are 13.7072$^\circ N$ and 79.5945$^\circ E$?

Basic Idea

- Interpolation means finding approximate values of an **unknown function** $f(x)$ for an x between different $x-$ values $x_0, x_1, \cdots, x_n$ at which the values of $f(x)$ are given.
- These values may come from mathematical function or measured data or automatically recorded values of an empirical function or from a table which are already recorded by researchers.
- For instance, air resistance of a car or an airplane at different speeds, yield of a chemical process at different temperatures, or the size of the size of India’s population appears from censuses taken at 10-year intervals.

Basic Idea

- Usually, we write these given values of a function in the form

$$f_0 = f(x_0), f_1 = f(x_1), \cdots, f_n = f(x_n)$$

or

$$(x_0, f_0), (x_1, f_1), \cdots, (x_n, f_n)$$

or

$$(x_i, f_i), i = 0, 1, \cdots, n$$

Basic Idea

- One can device any arbitrary function which satisfies these relations.
- However, polynomials are more convenient to work with as it can be easily differentiable, integrable, moreover, they are continuous.
- Also, they approximate continuous functions with desired accuracy which is given by the famous Weierstrass approximation theorem.

Polynomial Interpolation

Polynomial Interpolation

Theorem 1 (Weierstrass Approximation Theorem)

Polynomial Interpolation

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Let f be a continuous real valued function on $[a, b]$. Then, for each $\epsilon > 0$, there exists a polynomial $P_n(x)$ on $[a, b]$ such that

$$|f(x) - P_n(x)| < \epsilon, \quad \forall x \in [a, b]$$

Polynomial Interpolation

The general n-th order polynomial can be written as

$$P_n(x) = \sum_{i=0}^{n} a_k x^k \tag{1}$$

- For $n+1$ data points, there is a unique (why?) polynomial of order n that passes through all these points.
- For example, there is only one straight line that connects two points.
- There is only one parabola that connects a set of three points.
- Polynomial interpolation consists of determining a unique n-th order polynomial that fits $n+1$ data points.
- This polynomial provides a formula to compute intermediate values.

Polynomial Interpolation

For given values

$$(x_0, f_0), (x_1, f_1), \cdots (x_n, f_n)$$

we can find a polynomial $P_n(x)$ that satisfies the following condition:

$$P_n(x_i) = f_i, i = 0, 1, \cdots, n,$$

then P_n is called as an interpolation polynomial.
The next practical question is, to determine a_k. There are several methods available, we will examine two such methods, namely Lagrange interpolation and Newton's divided difference formula. The Taylor polynomial is also an option, but it has some shortcomings.

Polynomial Interpolation

Theorem 2 (Existence and Uniqueness theorem)
If $x_0, x_1, \cdots, x_n$ are distinct real numbers, then for arbitrary values of $f_0, f_1, \cdots, f_n$, there is a unique polynomial P_n of degree at most n such that

$$P_n(x_i) = f_i, 0 \leq i \leq n$$

Proof: The proof of existence is by induction. For $n = 0$, it is obvious as the constant function $P_0(x) = f_0$ satisfies. Now, if we have obtained a polynomial of degree P_k with

$$P_k(x_i) = f_i, 0 \leq i \leq k,$$

then we have to prove the result for $k + 1$.

Polynomial Interpolation

Let us construct a polynomial P_{k+1} in the following form,

$$P_{k+1}(x) = P_k(x) + c(x - x_0)(x - x_1)\cdots(x - x_k).$$

Obviously P_{k+1} is a polynomial and its degree is at most $k+1$. Since $P_k(x_i) = f_i, 0 \le i \le k$, we have

$$P_{k+1}(x_i) = P_k(x_i) = f_i, 0 \le i \le k$$

Polynomial Interpolation

Now, we have to find the unknown coefficient c from the given condition that $P_{k+1}(x_{k+1}) = f_{k+1}$. This yields the following equation.

$$P_k(x_{k+1}) + c(x_{k+1} - x_0)(x_{k+1} - x_1)\cdots(x_{k+1} - x_k) = f_{k+1}.$$

We can find a unique c from the above expression and hence a polynomial exists.

Polynomial Interpolation

For the proof of uniqueness, suppose there are two such polynomials, say P_n and Q_n. Then $W_n = P_n - Q_n$ is another polynomial with a property that $W_n(x_i) = 0, 0 \leq i \leq n$. Since W_n is a polynomial of degree at most n, this polynomial can have at most n roots. Since x_i's are distinct W_n has $n + 1$ roots. Therefore, W_n must be a zero polynomial. Therefore, $P_n = Q_n$.

Thanks

Doubts and Suggestions

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