MA633L-Numerical Analysis

Lecture 9 : Numerical Interpolation-Taylor Polynomials and Newton's Interpolating Polynomial

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January 23, 2025

Taylor Polynomials

Polynomial Interpolation

The general n-th order polynomial can be written as

$$P_n(x) = \sum_{k=0}^{n} a_k x^k \tag{1}$$

- For $n+1$ data points, there is a unique (why?) polynomial of order n that passes through all these points. (Existence and Uniqueness Theorem)
- Polynomial interpolation consists of determining a unique n-th order polynomial that fits $n+1$ data points.
- This polynomial provides a formula to compute intermediate values.

Taylor Polynomials

The Taylor polynomial for a $f \in C^n[a,b]$ about $x = x_0 \in [a,b]$ is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$$

- Taylor polynomials are one of the fundamental building blocks of numerical analysis.
- However, for polynomial approximation, this method can fail. Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate accuracy near that point only.
- A good approximating polynomial needs to provide a relative accuracy over an entire interval, whereas Taylor polynomials do not generally achieve this.

Taylor Polynomials

Example 1

Calculate the first six Taylor Polynomials about $x_0 = 0$ for $f(x) = e^x$.

Solution:

$$P_0(x) = 1$$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

Taylor Polynomials

The graph of this polynomial is given in the below figure.

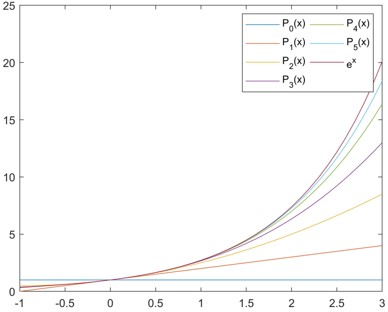

Figure 1: First Six Taylor Polynomials for e^x

Taylor Polynomials

Example 2

Calculate the first seven Taylor Polynomials about $x_0 = 1$ for $f(x) = 1/x$.

Solution:

$$f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}} \implies f^{(k)}(1) = (-1)^k k!$$

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-1)^k}{k!} f^{(k)}(1) \implies P_n(x) = \sum_{k=0}^{n} (1-x)^k$$

Taylor Polynomials

Now, $f(4) = 0.25$ is obvious, however, if we evaluate it using $P_n(4)$ for different n, we obtain the following table

n	0	1	2	3	4	5	6	7
$P_n(4)$	1	-2	7	-20	61	-182	547	-1640

You can observe that, none of the values are approaching nearer to $f(4)$. So, it is a big failure.
However $f(1.1) = 0.\overline{90}$ and $P_n(1.1)$ is very close enough.

n	0	1	2	3	4	5	6	7
$P_n(1.1)$	1	0.9000	0.9100	0.9090	0.9091	0.90909	0.909091	0.9090909

Taylor Polynomials

Remarks

1. For Taylor polynomials, all the information used in the approximation is concentrated at the single point x_0 so these polynomials generally give inaccurate approximations as we move away from x_0.
2. This limits Taylor polynomial approximation to situations in which approximations are needed only for values close to x_0.

Newton's Interpolating Polynomial

Newton's Interpolating Polynomial

Newton's interpolating polynomial is another approach to estimate the values of a_k. However, instead of obtaining a polynomial of the form (1) for given $n+1$ data points, $x_0, x_1, \cdots, x_n$ an n-th order polynomial is given by (usually, it is called as **Newton form of interpolation polynomial**).

$$P_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Therefore, our goal is to find b_k's instead of a_k's.

Linear Interpolation

When two points are given, as we do in school mathematics, the simplest way is to connect two data points using a straight line. This method is called linear interpolation. When $(x_0, y_0), (x_1, y_1)$ are two given points, we obtain the straight line equation as

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0}.$$

Let $y_0 = f(x_0) = f_0, y_1 = f(x_1) = f_1$ and $y = P_1(x)$, then we obtain that

$$\frac{P_1(x) - f_0}{f_1 - f_0} = \frac{x - x_0}{x_1 - x_0}.$$

Linear Interpolation

That is,

$$P_1(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0).$$

or

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad (2)$$

Equation (2) is called Newton's linear-interpolation formula. Here $P_1(x)$ denotes that this is a first-order polynomial.

Linear Interpolation

Remarks

- Notice that the term $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ represents the slope of the line connecting the two data points.
- Also, this term is a finite difference approximation of the first derivative.
- In general, smaller the interval, better the approximation, as the interval decreases, continuous function will be better approximated by straight line.
- This term is also referred as the first divided difference and denoted by $f[x_0, x_1]$

Linear Interpolation

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \tag{3}$$

where, $f[x_1]$ and $f[x_0]$ denote the zeroth divided difference of the function with respect to x_1 and x_0 respectively and defined as

$$f[x_0] = f(x_0) \quad \text{and} \quad f[x_1] = f(x_1) \tag{4}$$

Linear Interpolation

Hence, for first-order polynomial, we have

$$b_0 = f[x_0] \quad \text{and} \quad b_1 = f[x_0, x_1] \tag{5}$$

and, the Newton's linear interpolation formula is written as

$$P_1(x) = f[x_0] + f[x_0, x_1](x - x_0) \tag{6}$$

Linear Interpolation

Example 3

1. By means of Newton's linear interpolation formula, find the values of e^1 using the information that

x	0	2
e^x	1	7.3891

   Find the error ε_t.

2. Repeat the procedure using the following table

x	0	1.25
e^x	1	3.4903

Linear Interpolation

Solution for (1): First, let us calculate, $f[x_0], f[x_1], f[x_0, x_1]$

$$f[x_0] = f(x_0) = f(0) = 1$$
$$f[x_1] = f(x_1) = f(1) = 7.3891$$
$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{7.3891 - 1}{2} = 3.19455$$

Now, let us use these values in (6) to obtain $P_1(x)$.

$$\begin{aligned} P_1(x) &= f[x_0] + f[x_0, x_1](x - x_0) \\ &= 1 + 3.19455 * (x - 0) \\ &= 1 + 3.19455x \end{aligned}$$

Linear Interpolation

Now,

$$P_1(1) = 1 + 3.19455 = 4.19455$$

$$\varepsilon_t = \frac{|2.71828 - 4.19455|}{|2.71828|} = 0.5430$$

Solution for (2): Now, if we repeat the procedure for the second table

$$f[x_0] = f(x_0) = f(0) = 1$$
$$f[x_1] = f(x_1) = f(1.25) = 3.4903$$
$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{3.4903 - 1}{1.25} = 1.9922$$

Linear Interpolation

Now, let use these values in (6) to obtain $P_1(x)$.

$$
\begin{aligned}
P_1(x) &= f[x_0] + f[x_0, x_1](x - x_0) \\
&= 1 + 1.9922 * (x - 0) \\
&= 1 + 1.9922x
\end{aligned}
$$

Now,

$$P_1(1) = 1 + 1.9922 = 2.9922$$

$$\varepsilon_t = \frac{|2.71828 - 2.9922|}{|2.71828|} = 0.10078$$

We can see that shorter interval reduces the error. Also, the figure shows the deviation.

Linear Interpolation

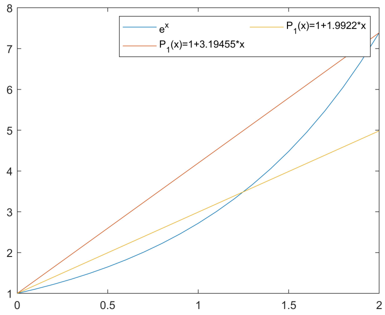

Figure 2: Newton Linear Interpolation

Newton's Interpolating Polynomial: Quadratic Interpolation

Quadratic Interpolation

In linear interpolation we obtained the polynomial as a linear function or first order polynomial or simply a straight line. When three points $(x_0, f_0), (x_1, f_1)$ and (x_2, f_2) are given, we can obtain a quadratic polynomial or introduce some curvature into the line connecting points or a parabola or second order polynomial. Conveniently, we write the following formula

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{7}$$

Quadratic Interpolation

Then, we can estimate b_0, b_1 and b_2 easily as follows: At $x = x_0$, $P_2(x_0) = f_0 = f[x_0]$, therefore,

$$b_0 = f[x_0].$$

At $x = x_1$, $P_2(x_1) = f_1 = f[x_1]$, therefore,

$$b_1 = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1].$$

At $x = x_2$, $P_2(x_2) = f_2 = f[x_2]$. Here, we are going to do some algebraic manipulations to obtain the value of b_2.

Quadratic Interpolation

$$b_2(x_2 - x_0)(x_2 - x_1) = f[x_2] - f[x_0] - \frac{f[x_1] - f[x_0]}{x_1 - x_0}(x_2 - x_0).$$

$$b_2(x_2 - x_0) = \frac{f[x_2] - f[x_0]}{x_2 - x_1} - \frac{f[x_1] - f[x_0]}{x_1 - x_0}\left(\frac{x_2 - x_0}{x_2 - x_1}\right).$$

$$b_2(x_2 - x_0) = \frac{f[x_2] - f[x_0] + f[x_1] - f[x_1]}{x_2 - x_1} - \frac{f[x_1] - f[x_0]}{x_1 - x_0}\left(\frac{x_2 - x_1 + x_1 - x_0}{x_2 - x_1}\right).$$

$$b_2(x_2 - x_0) = \frac{f[x_2] - f[x_1]}{x_2 - x_1} + \cancel{\frac{f[x_1] - f[x_0]}{x_2 - x_1}} - \frac{f[x_1] - f[x_0]}{x_1 - x_0} - \cancel{\frac{f[x_1] - f[x_0]}{x_2 - x_1}}$$

Quadratic Interpolation

Let

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

and

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0},$$

then

$$b_2 = f[x_0, x_1, x_2]$$

Therefore, the **Newton's quadratic interpolation polynomial** (7) is given by

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \qquad (8)$$

Quadratic Interpolation

Remarks

- Notice that

$$f[x_0, x_1, x_2] = f[x_1, x_2, x_0] = f[x_0, x_2, x_1] = f[x_1, x_0, x_2] = f[x_2, x_0, x_1] = f[x_2, x_1, x_0],$$

because, $f[x_0, x_1, x_2]$ is the coefficient of x^2 in the quadratic interpolation polynomial to interpolate f at x_0, x_1, x_2, whereas $f[x_1, x_2, x_0]$ is the coefficient of x^2 in the quadratic interpolation polynomial to interpolate f at x_1, x_2, x_0.

Quadratic Interpolation

Example 4

1. By means of Newton's quadratic interpolation formula, find the values of e^1 using the information that

x	0	2	4
e^x	1	7.3891	54.5981

   Find the error ε_t.

2. Repeat the procedure using the following table

x	0	1.25	1.5
e^x	1	3.4903	4.4817

Quadratic Interpolation

Solution for (1): First, let us calculate, $f[x_0], f[x_1], f[x_0, x_1], fx[x_0, x_1, x_2]$

$$f[x_0] = f(x_0) = f(0) = 1$$

$$f[x_1] = f(x_1) = f(1) = 7.3891$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{7.3891 - 1}{2} = 3.1946$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{54.5981 - 7.3891}{2} = 23.6045$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{23.6045 - 3.1946}{4} = 5.1025$$

Quadratic Interpolation

Now, let use these values in (8) to obtain $P_2(x)$.

$$\begin{aligned} P_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= 1 + 3.1946x + 5.1025x(x - 2) \\ &= 1 - 7.0104x + 5.1025x^2 \end{aligned}$$

Now,

$$P_2(1) = 1 + 3.1946 - 5.1025 = -0.9079$$

$$\varepsilon_t = \frac{|2.71828 + 0.9079|}{|2.71828|} = 1.3340$$

Quadratic Interpolation

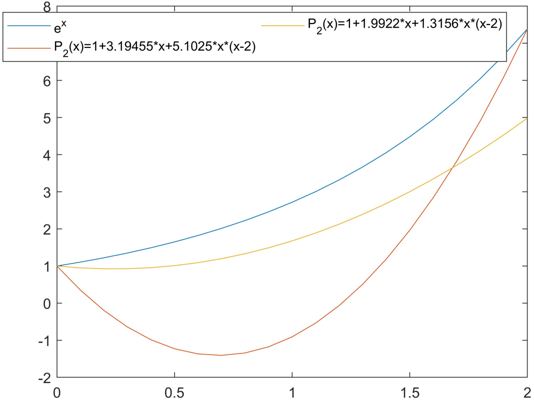

Figure 3: Newton Quadratic Interpolation

Quadratic Interpolation

Solution for (2): Now, if we repeat the procedure for the second table

$$f[x_0] = f(x_0) = f(0) = 1$$

$$f[x_1] = f(x_1) = f(1.25) = 3.4903$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{3.4903 - 1}{1.25} = 1.9922$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{4.4817 - 3.4903}{0.25} = 3.9656$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{3.9656 - 1.9922}{1.5} = 1.3156$$

Quadratic Interpolation

Now, let use these values in (8) to obtain $P_2(x)$.

$$
\begin{aligned}
P_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) \\
&= 1 + 1.9922 * (x - 0) + 1.3156x(x - 1.25) \\
&= 1 + 0.3477x + 1.3156x^2
\end{aligned}
$$

Now,

$$P_2(1) = 1 + 0.3477 + 1.3156 = 2.6633$$

$$\varepsilon_t = \frac{|2.71828 - 2.6633|}{|2.71828|} = 0.02021$$

We can see that shorter interval reduces the error.

Exercise

Exercise 1: Medium

1. Let $x_0 < x_1 < x_2$ be real, $f : \mathbb{R} \to \mathbb{R}$ be any function. If the unique polynomial $P_2(x) = a_0 + a_1 x + a_2 x^2$ passing through the data points $(x_0, y_0 = f(x_0)), (x_1, y_1 = f(x_1)), (x_2, y_2 = f(x_2))$, then prove that

$$a_2 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

and

$$f[x_0, x_1, x_2] = a_2$$

Thanks

Doubts and Suggestions

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